Generalized spectra and applications to finite distributive lattices

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Outline of the talk

- Motivation
- Topological frames
- $\mathbb{F}$-frames and $\mathbb{F}$-spectra
- Main Theorem for today
- Some consequences for free lattices

Motivation

- Functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ represents spaces as frames.
- Left adjoint $\Sigma : \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$ reconstructs the space:

$$\Sigma \mathcal{O} X \cong X \text{ whenever } X \text{ is sober.}$$

- $\mathbb{S} := (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$
- For any space X, $\mathcal{O}X \cong \mathrm{Top}(X, \mathbb{S})$.
- For any frame L,

$$\Sigma L = \mathbf{Frm}(L, \mathbf{2}) \cong \{a \in L \mid a \text{ is meet-irreducible}\}.$$

Motivation

- There are many other ways to construct frames induced by topological spaces, e.g. with $\mathbb{P} = ([0,\infty], \mathcal{T}_{\mathrm{Scott}})$

$$\mathbf{Top}(X, \mathbb{P})) = \{f : (X \to \mathbb{P}) \mid f \text{ continuous}\}.$$

- But the spectrum of $\mathbf{Top}(X, \mathbb{P})$ is $X \times]0, \infty]$, not X!
- Can we 'mod out' $]0, \infty]$?

Motivation

Question

How much/which topological information about a topological space X can be captured by the function frame $\mathrm{Top}(X, \mathbb{F})$ (endowed with pointwise order), and how does this depend on the nature of $\mathbb{F}$?

Topological frames

Definition

Let $\mathbb{F}$ be a frame endowed with a topology $\mathcal{T}_{\mathbb{F}}$. We call $(\mathbb{F}, \mathcal{T}_{\mathbb{F}})$ a *topological frame* provided that the operations

$$\wedge : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a, b) \mapsto a \wedge b$$

and

$$\sup_{i \in I} : \mathbb{F}^I \to \mathbb{F} : (a_i)_{i \in I} \mapsto \sup_{i \in I} a_i$$

are continuous.

E.g. any chain endowed with the Scott topology is a topological frame.

Topological frames

Definition

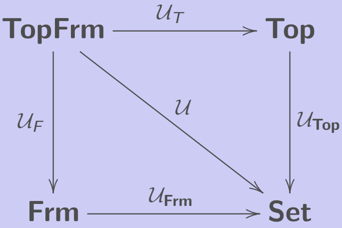
Let $(\mathbb{F}_1, \mathcal{T}_1)$ and $(\mathbb{F}_2, \mathcal{T}_2)$ be topological frames. A map $f : \mathbb{F}_1 \to \mathbb{F}_2$ is called a *topological frame morphism* if $f : (\mathbb{F}_1, \mathcal{T}_1) \to (\mathbb{F}_2, \mathcal{T}_2)$ is continuous and $f : \mathbb{F}_1 \to \mathbb{F}_2$ is a frame homomorphism.

We call **TopFrm** the category with topological frames as objects and topological frame morphisms as morphisms.

Topological frames

Proposition

The diagram



commutes, the functors $\mathcal{U}_{\mathbf{Top}}$ and $\mathcal{U}_F$ are topological, $\mathcal{U}_{\mathbf{Frm}}$ is monadic and has a left adjoint, $\mathcal{U}_T$ has a left adjoint and $\mathcal{U}$ is faithful and has a left adjoint.

$\mathbb{F}$-frames and $\mathbb{F}$-spectra

Let X be a topological space and $\mathbb{F}$ a topological frame.

- $\Gamma_X : \mathbb{F} \to \mathbf{Top}(X, \mathbb{F}) : a \mapsto c_a$ is a frame homomorphism
- Denote $\mathbf{Frm}_{\mathbb{F}}$ as the comma category $\mathbb{F}/\mathbf{Frm}$: $\mathbb{F}$-frames
- $\mathcal{O}_{\mathbb{F}} : \mathbf{Top} \to \mathbf{Frm}_{\mathbb{F}}^{\mathrm{op}}$ with $\mathcal{O}_{\mathbb{F}}(X) = \Gamma_X$ and

$$\mathcal{O}_{\mathbb{F}}(\varphi) : \mathbf{Top}(Y, \mathbb{F}) \to \mathbf{Top}(X, \mathbb{F}) : f \mapsto f\varphi$$

Let $L = (L, \gamma_L : \mathbb{F} \to L)$ be an $\mathbb{F}$-frame.

- Endow $\mathrm{Spec}_{\mathbb{F}}(L) = \mathbf{Frm}_{\mathbb{F}}(L, \mathbb{F})$ with the initial topology for the source

$$(\mathrm{ev}_l : \mathbf{Frm}_F(L, \mathbb{F}) \to \mathbb{F} : f \mapsto f(l))_{l \in L}$$

- We obtain a functor $\mathrm{Spec}_{\mathbb{F}} : \mathbf{Frm}_{\mathbb{F}}^{\mathrm{op}} \to \mathbf{Top}$ which is left adjoint to $\mathcal{O}_F$.

$\mathbb{F}$-spatial frames and $\mathbb{F}$-sober spaces

Definition

- L is $\mathbb{F}$-*spatial* if

$$\varepsilon_L : L \to \mathbf{Top}(\mathbf{Frm}_{\mathbb{F}}(L, \mathbb{F}), \mathbb{F}) : l \mapsto (f \mapsto f(l))$$

  is an isomorphism of $\mathbb{F}$-frames.
- X is $\mathbb{F}$-*sober* if

$$\eta_X : X \to \mathbf{Frm}_{\mathbb{F}}(\mathbf{Top}(X, \mathbb{F}), \mathbb{F}) : x \mapsto (f \mapsto f(x))$$

  is a homeomorphism.

If $\mathbb{F} = \mathbb{S}$, everything reduces to the classical setting.

More on $\mathbb{F}$-sobriety

proposition

X is $\mathbb{F}$-sober if and only if $(f : X \to \mathbb{F})_{f \in \mathbf{Top}(X,\mathbb{F})}$ is initial and point-separating, and $\mathbf{Frm}_{\mathbb{F}}(\mathbf{Top}(X,\mathbb{F}),\mathbb{F}) = \{\mathrm{ev}_x \mid x \in X\}$.

proposition

If X is Hausdorff and some conditions on $\mathbb{F}$ hold, then X is $\mathbb{F}$-sober.

Some examples

- $\mathrm{Spec}_{\mathbb{P}}(\mathbf{Top}(\mathbb{S}, \mathbb{P})) \simeq \mathbb{P}$
- $\mathrm{Spec}_{\mathbb{P}}(\mathbf{Top}(\mathbb{S}, \mathbf{n})) \simeq \mathbf{n}$
- $\mathrm{Spec}_{\mathbb{P}}(\mathbf{Top}(\mathbf{3}, \mathbb{P})) \simeq \{(\alpha, \beta) \in \mathbb{P} \times \mathbb{P} \mid \alpha \geq \beta\}$
- $\mathrm{Spec}_{\mathbf{n}}(\mathbf{Top}(\mathbf{3}, \mathbf{n})) \simeq \{(\alpha, \beta) \in \mathbf{n} \times \mathbf{n} \mid \alpha \geq \beta\}$
- $\mathrm{Spec}_{\mathbf{3}}(\mathbf{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3})) \simeq \mathbf{3} \times \mathbf{3}$
- $\ldots$

Some conlusions

Conclusion 1

A sober space is generally not $\mathbb{F}$-sober.

Conclusion 2

The space

$$\mathrm{Spec}_{\mathbb{F}}(L)$$

in general fails to be $\mathbb{F}$-sober.

Conclusion 3

The functor

$$\mathrm{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}} : \mathbf{Top} \to \mathbf{Top}$$

in general fails to be idempotent and does not give rise to an "$\mathbb{F}$-sobrification".

Relation to $\mathbb{F}-\textbf{Top}$

Consider the following adjunctions (D. Zhang and Y. Liu):

$$\textbf{Top} \underset{\iota_{\mathbb{F}}}{\overset{\omega_{\mathbb{F}}}{\underset{\bot}{\rightleftarrows}}} \mathbb{F}\text{-Top} \underset{\mathrm{pt}_{\mathbb{F}}}{\overset{\Omega_{\mathbb{F}}}{\underset{\bot}{\rightleftarrows}}} \textbf{Frm}_{\mathbb{F}}^{\mathrm{op}}$$

- $\omega_{\mathbb{F}} X$ is **F**-fuzzy sober + some condition on X implies that X is $\mathbb{F}$-sober,
- the notions differ,
- hence $\mathbb{F}$-sobriety also differs from fuzzy sobriety in the sense of A. Pultr and S. Rodabaugh.

Some questions

- For $\mathbb{F}_1, \mathbb{F}_2$ in a class of topological frames with additional properties, can we find a general description for $\mathrm{Spec}_{\mathbb{F}_2}(\mathbf{Top}(X, \mathbb{F}_1))$? Or just for $\mathbb{F}_1 = \mathbb{F}_2$?
- For what conditions on $\mathbb{F}_1, \mathbb{F}_2$ does

$$X \text{ Hausdorff} \Rightarrow X\ \mathbb{F}_1\text{-sober} \Rightarrow X\ \mathbb{F}_2\text{-sober} \Rightarrow X \text{ sober}$$

  hold?
- What can be said about the forgetful functors $\mathbf{Frm}_{\mathbb{F}} \to \mathbf{Frm}$ and $\mathbf{Frm}_F \to \mathbf{Set}$?

Main theorem for today

Some known facts about exponential objects:

- for X sober: X is exponential in **Top** $\Leftrightarrow X$ is locally compact,
- for X locally compact: the canonical topology on $\mathbf{Top}(X, Z)$ is the compact-open topology.

Theorem

Let X be a sober exponential topological space, $\mathbb{F}$ the topological frame $\mathbb{F} = \mathbf{Top}(Y, \mathbb{S})$ for some sober exponential space Y. Then $\mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X))$ and $\mathrm{Top}(Y, X)$ are homeomorphic, where the latter space carries the compact-open topology.

Main theorem

Sketch of proof:

- By the exponential law,

$$\mathbf{Top}(X, \mathbb{F}) = \mathbf{Top}(X, \mathbf{Top}(Y, \mathbb{S})) \simeq \mathbf{Top}(X \times Y, \mathbb{S})$$

  as spaces.
- Order taken to be pointwise and hence:

$$\mathbf{Top}(X, \mathbf{Top}(Y, \mathbb{S})) \cong \mathbf{Top}(X \times Y, \mathbb{S})$$

  as topological frames.

Main theorem

- It takes a bit of work to show that

  $$\mathcal{O} : \mathbf{Top}(Y, X \times Y) \to \mathbf{Frm}(\mathbf{Top}(X \times Y, \mathbb{S}), \mathrm{Top}(Y, \mathbb{S})) : f \mapsto \mathcal{O}(f)$$

  with $\mathcal{O}(f)(h) = hf$ is a bijection.
- Again using exponentiality, deduce that

  $$\mathbf{Top}(Y, X \times Y) \simeq \mathbf{Top}(Y, X) \times \mathbf{Top}(Y, Y).$$

- Show that the bottom line is a homeomorphism and restrict:

$$\begin{array}{ccc}
\mathbf{Top}(Y, X) & \longleftrightarrow & \mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X)) \\
\big\downarrow\hookrightarrow & & \big\downarrow\hookrightarrow \\
\mathbf{Top}(Y, X) \times \mathbf{Top}(Y, Y) & \longleftrightarrow & \mathbf{Frm}(\mathbf{Top}(X, \mathbb{F}), \mathbb{F})
\end{array}$$

Applications to $(\mathrm{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}})^n$

Lemma

Let X, Y be topological spaces with X finite. Then the compact-open topology on $\mathbf{Top}(X, Y)$ coincides with the initial topology for the source

$$(\mathrm{ev}_x : \mathbf{Top}(Y, X) \to Y)_{x \in X}.$$

Proposition

Let L, F be finite distributive lattices endowed with the Scott topology. Then $\mathbf{Top}(L, F) = \mathbf{Ord}(L, F)$ and the Scott topology and the initial topology on $\mathbf{Top}(L, F)$ determined by the source $(ev_l : \mathbf{Top}(L, F) \to F_l)_{l \in L}$, coincide.

Application 1: $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$

Definition

A monotone Boolean function of n variables is a non-decreasing map $\{0,1\}^n \to \{0,1\}$ where $\{0,1\}^n$ is endowed with the pointwise order. The set of monotone Boolean functions of n variables is denoted by M_n.

- Note that $M_n = \mathbf{Ord}(\mathbb{S}^n, \mathbb{S}) = \mathbf{Top}(\mathbb{S}^n, \mathbb{S})$.
- Since

$$\mathbf{Top}(\mathbf{1}, \mathbf{3}) = \mathbf{3} \to \mathrm{Top}(\mathbb{S}, \mathbb{S}) : \begin{cases} 0 \mapsto (0,0) \\ 1 \mapsto (0,1) \\ 2 \mapsto (1,1) \end{cases}$$

constitutes an isomorphism of posets and hence of spaces, we obtain that $\mathbf{3} \cong \mathbf{Top}(\mathbb{S}, \mathbb{S})$ as topological frames.

Application 1: $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$

Known fact

The lattice of monotone Boolean functions of n variables is isomorphic to the free distributive lattice on n generators.

Theorem

$(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$ and M_n are homeomorphic and hence isomorphic as lattices.

Application 1: $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$

Sketch of Proof:

- We will give a proof of the homeomorphism claim by induction.
- Applying Theorem our main theorem with $X = \mathbb{S}$, $\mathbb{F} = \mathbf{3} = \mathbf{Top}(\mathbb{S}, \mathbb{S})$ we can see that this is true for $n = 1$.
- Assume that our claim holds up to some $n - 1$ ($n \geq 2$). Applying our main theorem with $X = \mathbf{Top}(\mathbb{S}^{n-1}, \mathbb{S})$ and $\mathbb{F} = \mathbf{3} = \mathbf{Top}(\mathbb{S}, \mathbb{S})$, and the exponential law we get:

Application 1: $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$

$$\begin{aligned}
(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S}) &= \mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}}((\mathrm{Spec}_{\mathbf{3}}\mathcal{O})^{n-1}(\mathbb{S}) \\
&\simeq \mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}}(\mathbf{Top}(\mathbb{S}^{n-1}, \mathbb{S})) \\
&\simeq \mathbf{Top}(\mathbb{S}, \mathbf{Top}(\mathbb{S}^{n-1}, \mathbb{S})) \\
&\simeq \mathbf{Top}(\mathbb{S}^n, \mathbb{S}) \simeq M_n,
\end{aligned}$$

which finishes the proof of this part.

- Since all (finite) lattices in the proof are endowed with the Scott topology, the order isomorphism follows.

Application 2: $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$

- Denote the antichain on n elements by A_n and the free Boolean algebra on n elements by B_n.
- Consider $\mathbb{F} = \diamond = \mathbb{S} \times \mathbb{S} \cong \mathbf{Top}(A_2, \mathbb{S})$.

Lemma

$\mathbf{Top}(A_{2^{n-1}}, \diamond) \cong \mathbf{Ord}(A_{2^{n-1}}, \diamond) \cong B_n$.

Proof:
Since the discrete and the Scott topology on A_n coincide it follows that $\mathbf{Ord}(A_{2^{n-1}}, \diamond) = \diamond^{2^{n-1}}$.

Application 2: $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$

Theorem

$(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$ and B_n are homeomorphic and hence isomorphic as lattices.

Sketch of Proof:

- We will give a proof of the homeomorphism claim by induction.
- Applying Theorem our main theorem with $X = \mathbb{S}$, $\mathbb{F} = \diamond = \mathbf{Top}(A_2, \mathbb{S})$ we can see that

$$\mathrm{Spec}_\diamond \mathcal{O}_\diamond(\mathbb{S}) = \mathbf{Top}(A_{2^1}, \mathbb{S}) = \diamond = \mathbf{Top}(A_{2^0}, \diamond),$$

  so the claim is true for $n = 1$.
- Assume that our claim holds up to some $n-1$ $(n \geq 2)$. Applying our main theorem with $X = \mathbf{Top}(A_{2^{n-2}}, \diamond)$ and $\mathbb{F} = \diamond = \mathbf{Top}(A_2, \mathbb{S})$, and the exponential law we get:

Application 2: $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$

$$
\begin{aligned}
(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S}) &= \mathrm{Spec}_\diamond \mathcal{O}_\diamond((\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^{n-1}(\mathbb{S})) \\
&\simeq \mathrm{Spec}_\diamond \mathcal{O}_\diamond(\mathbf{Top}(A_{2^{n-2}}, \diamond)) \\
&\simeq \mathbf{Top}(A_2, \mathbf{Top}(A_{2^{n-2}}, \diamond)) \\
&\simeq \mathbf{Top}(A_{2^{n-1}}, \diamond) \simeq B_n,
\end{aligned}
$$

which finishes the proof of this part.

- Since all (finite) lattices in the proof are endowed with the Scott topology, the order isomorphism follows.

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Happy birthday Jorge!