Localic maps, dissolution and discrete covers

J. Picado and A. Pultr

presented by A. Pultr

To interpret frames and frame homomorphisms geometrically, that is, to have an extension of classical topology, on has to view the homomorphisms backwards, that is, to pass

from **Frm** to **Loc** = **Frm**$^{\mathrm{op}}$,

to the **category of locales**.

One can work simply abstractly with the dual category, but it is of advantage to work with **Loc** as with a concrete category. Frame homomorphisms

$$h : L \to M$$

can be uniquely represented by their right Galois adjoints

$$h_* : M \to L$$

and we speak of such maps as of *localic maps*.

Thus, what are the localic maps?

Being right adjoints, they preserve all meets. But not every map preserving all meets is *adjoint to a frame homomorphism*. Those that are are characterized by the conditions that

$$f(x) = 1 \quad \text{only for} \quad x = 1,$$

and

$$f(f^*(a) \to b) = a \to f(b)$$

the Frobenius identity ($\to$ is the Heyting arrow). Using thus defined maps turned out to be technically very useful, easier to work with then one would assume.

But do they make sense geometrically?

YES, they are more topological than one might expect.

Similarly as other categories, **Loc** has a natural concept of subobjects, namely the extremal monos (the extremal epis of **Frm**). As concrete subobjects they appear as the

sublocales S of L,

the subsets $S \subseteq L$ such that

- for every $M \subseteq S$ one has $\bigwedge M \in S$, and
- if $s \in S$ and $x \in L$ is arbitrary then $x \to s \in S$.

Thus defined concept of a sublocale naturally and aptly extends that of a subspace, and the system of all sublocales of L is a coframe

$$\mathsf{S}(L)$$

(Recall the co-Heyting (Boolean) algebra of all subspaces with the operation of difference $A \smallsetminus B$.)

Closed and open sublocales.

In particular we have the closed sublocales

$$\mathfrak{c}(a) = \uparrow a$$

and open sublocales

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\}$$

precisely corresponding to closed and open subspaces in the spatial case, and in general behaving precisely as closed and open subobjects should:

- $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other,
- meets of arbitrary many and joins of finitely many closeds are closed, and
- joins of arbitrary many and meets of finitely many opens are open,

etc.

Images and preimages under localic maps

For a localic $f : L \to M$, a sublocale $S \subseteq L$ and a sublocale $T \subseteq M$ we have the

image $f[S] \subseteq M$ and **preimage** $f_{-1}[T] \subseteq L$,

$f[S]$ being the standard set-theoretic image, and $f_{-1}[T]$ the largest sublocale contained in the standard $f^{-1}[T]$. They are adjoint to each other, that is,

$$f[A] \subseteq B \quad \text{iff} \quad A \subseteq f_{-1}[B],$$

Facts: 1. $f_{-1}[-]$ is a coframe homomorphism $\mathsf{S}(M) \to \mathsf{S}(L)$.

2. For **closed** A one has

$$f_{-1}[A] = f^{-1}[A]$$

but generally one has only $f_{-1}[A] \subseteq f^{-1}[A]$.

Perhaps not very surprisingly

Preimages of closed sublocales are closed sublocales and

but perhaps more surprisingly this can be inverted: roughly

If L, M are locales then a mapping $f : L \to M$ preserves closedness and openness by preimage then it is a localic map.

Why "roughly": standard set theoretic preimages $f^{-1}[-]$ are for open sublocales not always the same as $f_{-1}[-]$, hence we have to formulate it more correctly (S^c is the complement of S)

A map $f : L \to M$ between locales is localic iff

- for every closed $A \in \mathbf{S}(M)$, $f^{-1}[A]$ is closed, and

- for every open $A \in \mathbf{S}(M)$, $f^{-1}[A] \supseteq (f^{-1}[A^c])^c$.

Note that

- thus the localic maps are very much like continuous maps,

- but we have to think of the preimages of closed and open sublocales separately: We characterize localic maps among the *general ones* for which we can apply $f^{-1}[-]$ only, not the $f_{-1}[-]$ which would take care for complements.

We will now discuss some aspects of the frame (locale) $\mathsf{S}(L)^{\mathsf{op}}$ in view of the just presented facts.

Mainly we wish to explain what makes $\mathsf{S}(L)$ sometimes the more or less satisfactory surrogate for the classical discrete lifting of a space.

Discontinuous maps on a space X are studied as maps on $D(X)$, the discretization of X (the carrier provided with the discrete topology). In the study of discontinuity on a locale L one considers for a discretization the $\mathsf{S}(L)$ which is not really discrete, but being zero-dimensional (in a very strong way) it is indeed “dispersed enough” to help. But it may seem a somehow ad hoc technical way out. We will show that it really has to do with discreteness.

To this aim we will have to analyze the concept of *dissolution*.

Dissolutions were introduced by Isbell in the famous "Atomless parts of space" (Theorem 1.3), though not under this name. We follow the terminology of the excellent paper of Plewe where the author applies it to advantage. In Plewe's words, the dissolution of L is a (unique) locale L_a whose lattice of closed sets $\mathcal{C}\ell(L_a)$ is isomorphic to the lattice $\mathsf{S}(L)$ of all sublocales. hence we have here an isomorphism

$$\mathsf{S}(L) \cong \mathcal{C}\ell(L_a)$$

resulting in an embedding

$$\mathsf{S}(L) \cong \mathcal{C}\ell(L_a) \subseteq \mathsf{S}(L_a)$$

or, rather, an embedding

$$j : \mathsf{S}(L) \to \mathsf{S}(L_a)$$

such that $j[\mathsf{S}(L)]$ consists precisely of the closed sublocales of $\mathsf{S}(L_a)$.

In that the authors use any construction of subobjects then available (onto maps, nuclei, congruences) and Isbell's construction is based on a general inverse-image (preimage) procedure working in any complete category.

In fact the situation is much simpler and more transparent. From now on sublocales are the subsets mentioned above, and the same with images and preimages. To avoid repeated explanation of when we speak of frames and when on coframes we will write

$$\mathsf{T}(L) \quad \text{for} \quad \mathsf{S}(L)^{\mathsf{op}}.$$

and remember the standard embedding

$$\mathfrak{c}_L = (a \to \uparrow a) : L \to \mathsf{T}(L).$$

We can continue in constructing *assembly (tower)*

$$L \xrightarrow{\mathfrak{c}_L} \mathsf{T}(L) \xrightarrow{\mathfrak{c}_{\mathsf{T}(L)}} \mathsf{T}(\mathsf{T}(L)) \longrightarrow \cdots$$

$\mathfrak{c}_L$ is a frame homomorphism, hence a left adjoint to a localic map γ_L

$$L \underset{\gamma_L}{\overset{\mathfrak{c}_L}{\rightleftarrows}} \mathsf{T}(L) \quad (\mathfrak{c}_L \dashv \gamma_L)$$

and this γ_L has a very simple formula

$$\gamma_L(S) = \bigwedge S.$$

and we have the tower in more detail as

$$L \underset{\gamma_L}{\overset{\mathfrak{c}_L}{\rightleftarrows}} \mathsf{T}(L) \underset{\gamma_L}{\overset{\mathfrak{c}_L}{\rightleftarrows}} \mathsf{T}(\mathsf{T}(L)) \longrightarrow \cdots$$

Now let us concentrate to two things. First we have the image-preimage adjunction for the (very simple) localic map γ_L

$$\mathsf{T}(L) \underset{\gamma_L[-]}{\overset{(\gamma_L)_{-1}[-]}{\rightleftarrows}} \mathsf{T}(\mathsf{T}(L)) \,. \quad \perp$$

But we have already seen another adjunction pair between the $\mathsf{T}(L)$ and $\mathsf{T}\mathsf{T}(L)$, namely the second step

$$\mathsf{T}(L) \underset{\gamma_{\mathsf{T}(L)}}{\overset{\mathfrak{c}_{\mathsf{T}(L)}}{\rightleftarrows}} \perp \ \mathsf{T}(\mathsf{T}(L))$$

of the assembly above. And it needs only a few lines of easy computing to show that they coincide. The embedding $\mathfrak{c}_{\mathsf{T}(L)}$ of course embeds $\mathsf{T}(L)$ precisely on the $C\ell(\mathsf{T}(L))$ and hence so does the $(\gamma_L)_{-1}[-]$.

Thus, the dissolution is nothing else but

$$L_a = \mathsf{T}(L) = \mathsf{S}(L)^{\mathsf{op}}$$

Let us now compare the classical discrete lifting in **Top**, $\delta : D(X) \to X$, with the localic map $\gamma : \mathsf{T}(L) \to L$ in **Loc**.

For the δ we have that

> is is one-to-one onto, and $\delta^{-1}[A]$ is closed for each subset $A \subseteq X$

For the $\gamma : \mathsf{T}(L) \to L$ we have that

> it is monic and epic, and $\gamma_{-1}[A]$ is closed for each sublocale $A \subseteq L$.

It does look very much the same !

But there is, of course, a difference.

In the former, the condition on closed sets implies the same for open sets.

In the latter, the open sets are not attended to.

THUS: **using the S**(L) **is a sort of discretization**,

BUT it goes only

a half way to a full discretization.

It has to be noted that for some purposes it may be the best what we can have.

In the following diagram, $\mathsf{S}_{\mathsf{c}}(L)$ is Boolean and hence "point-free discrete".

$\mathfrak{B}$ is the Booleanization, and B the maximal essential extension.

The dashed arrows indicate that those make sense in the subfit case only.