A Day on Pointfree Topology

University of Coimbra

October 12, 2023

Abstracts

The geometry of locale maps via Galois adjunctions

João Areias

Centre for Mathematics, University of Coimbra

We will look at four adjoint situations in pointfree topology that interchange images and preimages with closure and interior operators, and establish with them a number of characterizations for meet-preserving maps, localic maps, open maps (in a broad sense) and open localic maps between locales. In order to do this we will need to generalize the interior and closure of sublocales to general subsets. This will be done using just elementary ideas and basic concepts of localic topology.

Revisiting localic T_1-type separation

Igor Arrieta

School of Computer Science, University of Birmingham

There are a number of localic separation axioms which are roughly analogous to the T_1-axiom from classical topology. For instance, besides the well-known subfitness and fitness, there are also Paseka-Šmarda's T_1-locales, totally unordered locales, and the recently introduced $\mathcal{F}$-separated locales (i.e., those which are separated w.r.t. the fitting closure operator in Loc) — a property strictly weaker than fitness.

In this talk, I will compare $\mathcal{F}$-separatedness with other T_1-type axioms, and explain how this property is, in a certain sense, dual to Isbell's strong Hausdorff property. We will also look at $\mathcal{F}$-compact objects (i.e., those which are categorically compact w.r.t. the fitting closure operator in Loc), showing that the situation differs radically from the category of topological spaces.

Joint work with Jorge Picado and Aleš Pultr.

Measurable functions on σ-frames

Raquel Bernardes

Centre for Mathematics, University of Coimbra

In this talk, we will study (semi)measurable functions on σ-frames, extending the theory of real-valued functions from frames to σ-frames. Our objects of study are the σ-frame homomorphisms $\mathfrak{L}(\mathbb{R}) \longrightarrow \mathfrak{C}(L)$ from the usual frame of reals into the congruence lattice of a σ-frame L, and its subclass of measurable functions $\mathfrak{L}(\mathbb{R}) \longrightarrow L$.

We will show that, despite σ-frames not being pseudocomplemented, we can still present scaling methods for generating measurable functions akin to the ones used for real-valued functions on frames, using the notions of σ-scale and finite σ-scale (which, in the presence of pseudocomplements, reduce to the notions of extended scale and scale in frames).

Moreover, we will address the problem of inserting measurable real functions in between more general functions on σ-frames and deduce extension and separation results also valid for a general σ-frame. A characterisation of several classes of σ-frames (normal, extremally disconnected, $\mathcal{G}$-perfect, $\mathcal{F}$-perfect and perfectly normal) in terms of insertion, extension and separation conditions also holds, extending the familiar results for real-valued functions on frames to measurable and semimeasurable functions.

Extending Stone-Priestley duality along full embeddings

Célia Borlido

Centre for Mathematics, University of Coimbra

Stone and Priestley dualities for bounded distributive lattices establish dual equivalences with the categories of spectral spaces and of Priestley spaces, respectively (and these two categories are isomorphic). Since bounded distributive lattices are, in turn, equivalent to coherent frames, these dualities may be seen as suitable restrictions and co-restrictions of the spatial-sober duality between spatial frames and sober spaces. Notice however that neither spectral nor Priestley spaces are fully embedded in the category of sober spaces: morphisms of spectral spaces are required to be well-behaved with respect to compactness, while Priestley spaces come equipped with an order, and their morphisms are required to be order-preserving. Similarly, coherent frames do not form a full subcategory of frames as the morphisms are required to preserve compact elements. In the past few years, several authors suggested it could be more natural to understand dualities for bounded distributive lattices bitopologically [6, 5, 1]. Unlike in the monotopological setting, Stone-Priestley duality may be seen as a restriction and co-restriction of the bitopological version of the spatial-sober duality along full subcategory embeddings.

In this talk we will see an alternative extension of Stone-Priestley duality along full embeddings to a spatial-sober-like adjunction. The categories at play are those of *Pervin spaces* and of *Frith frames*. Pervin spaces, introduced in [4], are known to capture totally bounded and transitive quasi-uniform spaces. Frith frames are its pointfree version, and we show they form a full coreflective subcategory of totally bounded and transitive quasi-uniform frames. In particular, there exist natural notions of *completion of a Pervin space* and of *completion of a Frith frame* available. It may be derived from [7] that the category of complete T_0 Pervin spaces is equivalent to the category of spectral topological spaces, and we show that complete Frith frames are equivalent to coherent frames. We will also see how this relates to the bitopological point of view mentioned above.

This is based on joint work with Anna Laura Suarez [3, 2].

References

[1] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz. Bitopological duality for distributive lattices and Heyting algebras. *Mathematical Structures in Computer Science*, 20(3):359–393, 2010.

[2] C. Borlido and A. L. Suarez. Pervin spaces and Frith frames: bitopological aspects and completion. Accepted for publication in *Applied Categorical Structures* (preprint available at `https://arxiv.org/abs/2303.00443`), 2023.

[3] C. Borlido and A. L. Suarez. A pointfree theory of pervin spaces. *Quaestiones Mathematicae*, 0(0):1–40, 2023.

[4] M. Gehrke, S. Grigorieff, and J.-É. Pin. A topological approach to recognition. In *Automata, languages and programming. Part II*, volume 6199 of *Lecture Notes in Comput. Sci.*, pages 151–162. Springer, Berlin, 2010.

[5] A. Jung and M. A. Moshier. On the bitopological nature of Stone duality. Technical Report CSR-06-13, University of Birmingham, 2006. 110 pages.

[6] J. Picado. Join-continuous frames, Priestley's duality and biframes. *Appl. Categ. Structures*, 2(3):297–313, 1994.

[7] J.-É. Pin. Dual space of a lattice as the completion of a pervin space. In *International Conference on Relational and Algebraic Methods in Computer Science*, pages 24–40, 04 2017.

Sequences suffice for pointfree completions

Graham Manuell

Centre for Mathematics, University of Coimbra

Completions of metric spaces are usually constructed using Cauchy sequences. However, this does not work for general uniform spaces, where Cauchy filters or nets must be used instead. The situation in pointfree topology is more straightforward: the correct completion of uniform locales can indeed be obtained as a quotient of a locale of Cauchy sequences.

Localic maps, dissolution, and discrete covers

Aleš Pultr

Charles University

To interpret theory of frames geometrically, that is, as a point-free extension of classical topology, one has to think, for obvious reasons, in terms of the *dual* $\mathbf{Loc} = \mathbf{Frm}^{\mathsf{op}}$ of the category of frames. It is of advantage to represent **Loc** as a concrete category with the "inverse arrows" of frame homomorphisms $h : A \to B$ represented by their right Galois adjoints $f : B \to A$. This turned out to be more than just a technically usefull ad hoc representation. Frames (locales) have well defined closed and open subobjects (sublocales), and those are, as in classical topology, preserved by (well defined) preimages by localic maps. This is not much of a surprise, but one has more: quite like classical continuous maps, locally maps are *precisely the standard maps preserving closed and open sublocales by preimages* (how one does it with the standard preimages has to be explained, but it will be done).

The dual $\mathsf{T}(L)$ of the coframe $\mathsf{S}(L)$ of sublocales of L is a natural extension of L (one sends a to the closed sublocale $\uparrow a \subseteq L$). It is not Boolean, that is, it is not "point-free discrete" but it is dispersed enough to be used for some purposes similarly like the discretisation of a classical space is. The result on localic map above, also in view of the concept of Isbell's and Plewe's dispersion shows that this approach to discontinuity is not just an ad hoc surrogate of a missing "really discrete cover" but a natural step towards discreteness. Indeed, while the classical discrete lifting $\delta : D(X) \to X$

is one-to-one onto, and $\delta^{-1}[A]$ is closed for each subset $A \subseteq X$,

the cover $\gamma : \mathsf{T}(L) \to L$ in **Loc**

is monic and epic, and $\gamma_{-1}[A]$ is closed for each sublocale $A \subseteq L$.

It should be noted that, however, unlike the former, the latter goes to the discreteness only half way: in the classical space the statement on closedness implies the same on openness, while in the latter case a statement on openness is not implied.

If there will be time we can add some information on a "really discrete cover" which, however, cannot replace $\mathsf{S}(L)$ in everything.

Joint work with Jorge Picado.

Generalized spectra and applications to finite distributive lattices

Mark Sioen

Free University of Brussels VUB

This talk reports on joint work with W. Lowen and W. Van Den Haute. In previous joint work, the classical adjunction between topological spaces and frames was generalized to a setup in which an arbitrary topological frame replaces the two element chain. The relevant composition of adjoints yields an endofunctor on topological spaces which in general fails to be idempotent. In this paper we prove a formula for iterations of this functor under certain conditions. We apply our result to the construction of finite free distributive lattices and Boolean algebras.

Pointfree bitopological subspaces

Anna Laura Suarez

University of Padova

In the literature, there exist two pointfree versions of the concept of bitopological space: biframes as in [1] and d-frames as in [2]. We discuss the advantages and disadvantages of both theories and address the following open problem: how does one obtain a satisfactory pointfree notion of bitopological subspace? We illustrate how the notion of finitary biframe solves some of the disadvantages of the two theories.

References

[1] B. Banaschewski, G. C. L. Brümmer and K. A. Hardie, Biframes and Bispaces, *Quaestiones Mathematicae*, 6:1-3, 13–25, 1983.

[2] A. Jung and M. A. Moshier. On the bitopological nature of Stone duality. Technical Report CSR-06-13, University of Birmingham, 2006. 110 pages.