Pointfree bispaces, pointfree bisubspaces

On the connection between sobriety and the system of subspaces in the bitopological setting

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Overview

The system of all subspaces: pointfree setting

We start by recalling an important fact about the system of all sublocales of a locale.

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For a frame L, the ordered collection $\mathsf{S}(L)^{op}$ is a frame.

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This suggests that the sublocales of a frame may be interpreted as closed sets of some topology.

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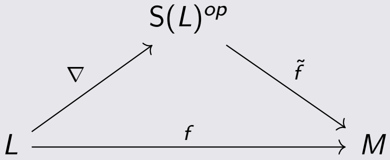
We have a canonical embedding $\nabla : L \hookrightarrow \mathsf{S}(L)^{op}$.

Theorem (Joyal and Tierney, 1984)

For a frame L, the embedding $\nabla : L \hookrightarrow \mathsf{S}(L)^{op}$ is such that for every frame map $f : L \to M$ such that every $f(x)$ has a complement there is $\tilde{f} : \mathsf{S}(L)^{op} \to M$ making the following diagram commute.



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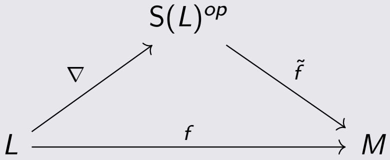

This is what we mean when we say that $\mathsf{S}(L)^{op}$ *provides complements freely* to the elements of L.

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For a sober space X, its sober subspaces are closed under arbitrary intersections and finite unions.

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For a topological space X, the *Skula topology* on its points is the one generated by the opens of X together with their complements.

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Theorem (Keimel and Lawson, 2009)

The sober subspaces of a sober space are the closed sets of the Skula topology.

The system of all subspaces: comparing the two settings

The frame of sublocales of a frame is then a pointfree analogue of the Skula topology of a space. In what precise sense is it its analogue? The following result is well-known, see for instance Picado and Pultr's book [4].

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For a frame L, we have that $\mathrm{pt}(\mathsf{S}(L)^{op})$ is – up to homeomorphism – the Skula space of $\mathrm{pt}(L)$.

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For a frame L, we have that $\mathrm{pt}(\mathsf{S}(L)^{op})$ is – up to homeomorphism – the Skula space of $\mathrm{pt}(L)$. In particular, the assignments $L \mapsto \mathsf{S}(L)^{op}$ and $X \mapsto Sk(X)$ are functorial and the following commutes up to natural isomorphism.

$$\begin{array}{ccc} \mathbf{Frm}^{op} & \xrightarrow{\mathsf{S}^{op}} & \mathbf{Frm}^{op} \\ \downarrow{\scriptstyle \mathrm{pt}} & & \downarrow{\scriptstyle \mathrm{pt}} \\ \mathbf{Top} & \xrightarrow{Sk} & \mathbf{Top} \end{array}$$

The system of all subspaces: comparing the two settings

Let us review the importance of the pointfree and point-set versions of the system of all subspaces.

For a frame L...

- The system of all its sublocales is the opposite of a frame.
- This is $\mathsf{S}(L)^{op}$, the *frame of sublocales*.
- It provides complements freely to all elements of the original frame.

For a sober space X...

- The system of all its sober subspaces are the closed sets of some topology.
- This is called the *Skula topology* on X.
- It is generated by adding the closed sets to the original topology.

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- It is generated by adding the closed sets to the original topology.

We call the facts on the left **UP** (for *universal property*) and those on the right **Sob** (for *sobriety*). We will now seek for a good bitopological version of both **UP** and **Sob**.

Bitopological spaces

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Bitopological spaces arise naturally when dealing with *quasi-uniform* spaces. They also provide a good setting in which to speak about Stone-type dualities (see Jung and Moshier 2006 [3], Bezhanishvili et al. 2010 [2]).

Biframes and d-frames

There are two pointfree analogues of bitopological spaces. We have *biframes*, studied in Banaschewski (see [1]). More recently, *d-frames* have been introduced by Jung and Moshier in 2006 (see [3]). Let us quickly look at their dualities.

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We have an adjunction $\mathrm{b}\Omega : \mathbf{biTop} \leftrightarrows \mathbf{biFrm}^{op} : \mathrm{bpt}$ *with* $\mathrm{b}\Omega \dashv \mathrm{bpt}$. *The functor* $\mathrm{b}\Omega$ *assigns to each bitopological space* X *the triple* $(\tau^+, \tau^-, < \tau^+ \cup \tau^- >)$

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We observe that this functor keeps all the information on the patch topology.

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Intuitively, the functor $\mathrm{d}\Omega$ only keeps the information on which pairs of $\tau^+ \times \tau^-$ are disjoint and which are covering.

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Intuitively, $\mathfrak{c} \circ \mathrm{b}\Omega$ is only keeping the information on the order relations between the *finitary* elements of $< \tau^+ \cup \tau^- >$.

Finitary biframes

The adjunction $\mathfrak{c} \circ \mathrm{b}\Omega : \textbf{biTop} \leftrightarrows {\textbf{biFrm}_{\textbf{fin}}}^{op} : \mathrm{bpt}$ is a middle ground between the biframe and the d-frame adjunctions. The "open set" functor of this adjunction forgets more information than the biframe one, but less than the d-frame one.

$$\textbf{BiTop} \underset{\mathrm{bpt}}{\overset{\mathrm{b}\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \textbf{BiFrm}^{op} \underset{}{\overset{\mathfrak{c}}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\bot}}}} {\textbf{BiFrm}_{\textbf{fin}}}^{op} \underset{\Gamma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \textbf{dFrm}^{op}$$

Biframes and d-frames

The dualities of biframes and d-frames give raise to different bitopological versions of sobriety. It turns out that the fixpoints of the biframe adjunction are simply the *patch-sober* ones. We call *d-sober* bispaces the fixpoints of the d-frame adjunction. Both these notions present some issues.

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This means that we cannot hope to have a satisfactory bitopological version of **Sob** in these two settings.

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We say that a bispace is *bisober* when it is a fixpoint for the finitary biframes adjunction.

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We say that a bispace is *bisober* when it is a fixpoint for the finitary biframes adjunction. For a bitopological space X, the *Skula bitopology* on its points is the one

- whose positive opens are the topology generated by τ^+ together with the complements of the opens in τ^-;
- whose negative opens are the topology generated by τ^- together with the complements of the opens in τ^+.

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The bisober bisubspaces of a bisober bispace are the patch-closed sets of the Skula bitopology.

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We have then found a bitopological version of **Sob** in the setting of finitary biframes.

The congruence biframe

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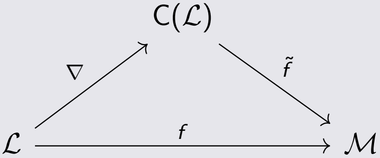
Theorem (Schauerte 1992)

For every biframe $\mathcal{L}$ there is a biframe $\mathsf{C}(\mathcal{L})$ together with a biframe embedding $\nabla : \mathcal{L} \to \mathsf{C}(\mathcal{L})$ such that whenever $f : \mathcal{L} \to \mathcal{M}$ is a biframe map providing bicomplements to all elements of L^+ and L^-, there is $\tilde{f} : \mathsf{C}(\mathcal{L}) \to \mathcal{M}$ such that the following commutes.



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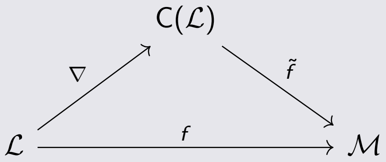

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- The structure $\mathsf{C}(\mathcal{L})$ does not represent the bisublocales of $\mathcal{L}$. Furthermore, the biframe version of bisublocale collapses to the monotopological notion for its patch (just like with sobriety).

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This means that we cannot find a satisfactory bitopological version of **UP** in the settings of biframes or d-frames.

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In [6] the following is proven.

Lemma (Suarez, 2022)

For a finitary biframe $\mathcal{L}$ the biframe $\mathsf{C}(\mathcal{L})$ is finitary, and it provides bicomplements freely to $\mathcal{L}$ in the category $\mathbf{biFrm}_{fin}$.

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For a finitary biframe $\mathcal{L}$ the biframe $\mathsf{C}(\mathcal{L})$ is finitary, and it provides bicomplements freely to $\mathcal{L}$ in the category $\mathbf{biFrm}_{fin}$. Furthermore, the main component of the biframe $\mathsf{C}(\mathcal{L})$ is anti-isomorphic to the ordered collection of the its finitary bisublocales.

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For a finitary biframe $\mathcal{L}$ the biframe $\mathsf{C}(\mathcal{L})$ is finitary, and it provides bicomplements freely to $\mathcal{L}$ in the category $\mathbf{biFrm}_{fin}$. Furthermore, the main component of the biframe $\mathsf{C}(\mathcal{L})$ is anti-isomorphic to the ordered collection of the its finitary bisublocales.

We have then found a bitopological version of **UP** in the setting of finitary biframes.

The system of all bisubspaces: comparing the two settings

From Schauerte's result and the definition of Skula bispace, it is easy to prove the following.

Theorem

For a finitary biframe $\mathcal{L}$, we have that $\mathrm{bpt}(\mathsf{C}(\mathcal{L}))$ *is – up to bihomeomorphism – the Skula bispace of* $\mathrm{bpt}(\mathcal{L})$.

The system of all bisubspaces: comparing the two settings

From Schauerte's result and the definition of Skula bispace, it is easy to prove the following.

Theorem

For a finitary biframe $\mathcal{L}$, we have that $\mathsf{bpt}(\mathsf{C}(\mathcal{L}))$ *is – up to bihomeomorphism – the Skula bispace of* $\mathsf{bpt}(\mathcal{L})$. *In particular, the assignments* $\mathcal{L} \mapsto \mathsf{C}(\mathcal{L})$ *and* $X \mapsto biSk(X)$ *are functorial and the following commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{biFrm_{fin}}^{op} & \xrightarrow{\mathsf{C}} & \mathbf{BiFrm_{fin}}^{op} \\ \Big\downarrow \mathsf{bpt} & & \Big\downarrow \mathsf{bpt} \\ \mathbf{biTop} & \xrightarrow{biSk} & \mathbf{biTop} \end{array}$$

The system of all bisubspaces: comparing the two settings

We have obtained bitopological versions of **UP** and **Sob**.

For a finitary biframe $\mathcal{L}$...

- The system of all its bisublocales forms a biframe.
- This is $\mathrm{C}(\mathcal{L})$, Schauerte's congruence biframe.
- It provides bicomplements freely to all elements of $L^+ \cup L^-$.

For a bisober bispace X...

- The system of all its bisober bisubspaces are the closed sets of some bispace.
- This is called the *Skula bitopology* on X.
- It is generated by adding the closed sets of one topology to the opens of the other.

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