Sequences suffice for pointfree completions

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Uniform spaces

A uniform space is a set X equipped with a filter $\mathcal{E}$ of binary relations on X satisfying, for all $E \in \mathcal{E}$:

1. $\Delta_X \subseteq E$,
2. $E^o \in \mathcal{E}$,
3. $\exists F \in \mathcal{E}.\ F \circ F \subseteq E$.

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A map $f\colon X \to Y$ between uniform spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ is uniformly continuous if $(f \times f)^{-1}(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$.

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Define $v \triangleleft^E u$ (for $u, v \in \mathcal{O}X$, $E \in \mathcal{E}$) to mean $E \circ (v \oplus v) \leq u \oplus u$ and $v \triangleleft^{\mathcal{E}} u$ to mean $v \triangleleft^E u$ for some $E \in \mathcal{E}$.

A pre-uniform locale $(X, \mathcal{E})$ is a uniform locale if $u = \bigvee_{v \triangleleft^{\mathcal{E}} u} v$ for all u.

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A locale map $f\colon X \to Y$ between pre-uniform locales $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ is said to be uniform if $(f \times f)^*(F) \in \mathcal{E}$ for all $F \in \mathcal{F}$.

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A uniform embedding is a sublocale embedding $f \colon X \hookrightarrow Y$ such that $(f \times f)^*(F) \in \mathcal{E}$ if and only if $F \in \mathcal{F}$.

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The completion of a uniform space/locale is usually constructed in terms of (regular) Cauchy filters.

A regular Cauchy filter on a uniform locale $(X, \mathcal{E})$ is a filter F on $\mathcal{O}X$ such that

- F is nontrivial in the sense that $u \in F \implies u > 0$,
- for each $E \in \mathcal{E}$, there is some $u \in F$ with $u \oplus u \leq E$,
- if $u \in F$ then there is a $v \in F$ such that $v \triangleleft^{\mathcal{E}} u$.

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There is an obvious locale embedding $\gamma \colon X \hookrightarrow \mathcal{C}X$ obtained by sending $[u \in F] \in \mathcal{O}\mathcal{C}X$ to $u \in \mathcal{O}X$. This is the completion of X.

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Vickers showed that the underlying locale of the completion of a metric *space* can be obtained as a (tri)quotient of the locale of rapidly converging Cauchy sequences.[1]

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We will use a similar approach (though there is no good way to define rapid convergence without a metric).

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Modulated Cauchy sequences

A Cauchy sequence in a uniform space $(X, \mathcal{E})$ is a map $s \colon \mathbb{N} \to X$ such that $\forall E \in \mathcal{B}.\ \exists N \in \mathbb{N}.\ \forall n, n' \geq N.\ (s(n), s(n')) \in E$ where $\mathcal{B}$ is some chosen base for the uniformity $\mathcal{E}$.

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We can now define a locale of modulated Cauchy sequences.

The locale of modulated Cauchy sequences

Let $X = (X, \mathcal{E})$ be a uniform locale with base $\mathcal{B} \subseteq \mathcal{E}$. We give a presentation for the frame of $\mathrm{ModCauchy}(X)$.

The generators are:

- $[s(n) \in u]$ for each $n \in \mathbb{N}$ and $u \in \mathcal{O}X$,
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The relations are:

- $\bigvee_\alpha \bigwedge_{u \in F_\alpha} [s(n) \in u] = [s(n) \in \bigvee_\alpha \bigwedge F_\alpha]$ for each family $(F_\alpha)_\alpha$ of finite subsets of $\mathcal{O}X$,
- $1 \leq \bigvee_{k \in \mathbb{N}} [m(E) = k]$ for each $E \in \mathcal{B}$,
- $[m(E) = k] \leq \bigvee_{u \oplus u' \leq E} [s(n) \in u] \wedge [s(n') \in u']$ for $E \in \mathcal{B}$, $k \in \mathbb{N}$ and $n, n' \geq k$.

The limit map

We will now argue that the completion of X is a well-behaved quotient of $\mathrm{ModCauchy}(X)$.

The quotient map $q\colon \mathrm{ModCauchy}(X) \to \mathcal{C}X$ is intended to 'take the limit' of the Cauchy sequences and is given by

$$q^*([u \in F]) = \bigvee_{E \in \mathcal{B}} \bigvee_{v \triangleleft^E u' \triangleleft^{\mathcal{E}} u} \bigvee_{k \leq k' \in \mathbb{N}} [m(E) = k] \wedge [s(k') \in v].$$

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Intuitively, this says $q((m, s))$ lies in u iff $s(k') \in v$ for some $k' \in \mathbb{N}$ and $v \lhd^E u' \lhd^{\mathcal{E}} u$ such that $m(E) = k$ and $k \leq k'$.

The triquotient assignment

To show this is a ‘good’ quotient, we will define a join-preserving map $g\colon \mathcal{O}\mathrm{ModCauchy}(X) \to \mathcal{O}\mathcal{C}X$ such that $gq^* = \mathrm{id}_{\mathcal{O}\mathcal{C}}$ and for all a, b we have $g(a \wedge q^*(b)) = g(a) \wedge b$.

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I won’t give the definition g here, but the intuition is that $\overline{g}(p)$ is the collection of modulated Cauchy sequences (m, s) that converge to p ‘twice as fast’ as the modulus.

Taking stock

Note that we can now throw away our old construction of $\mathcal{C}X$ and recover it from the fixed points of the composite endofunction q^*g on $\mathcal{O}\mathrm{ModCauchy}(X)$.

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Well, the spatial approach would involve restricting to the points of $\mathrm{ModCauchy}(X)$ *before* we take the quotient. And the spatial coreflection does *not* commute with the quotient.

Moreover, note that $\mathrm{ModCauchy}(X)$ can be highly non-spatial when $\mathcal{B}$ is uncountable. Intuition: $\mathbb{N}^{\mathcal{B}}$ is non-spatial for uncountable $\mathcal{B}$.