Extending Stone duality along full embeddings

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based on joint work with A. L. Suarez

Centre for Mathematics, University of Coimbra

A day on Pointfree Topology

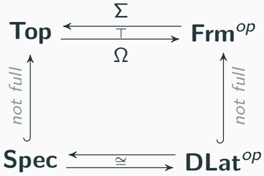
Celebrating Jorge Picado's 60th birthday

Coimbra, October 12, 2023

1. The spatial-sober duality

   (and its restriction to Stone duality for bounded distributive lattices)

2. The categories of Pervin spaces and of Frith frames
3. Extending Stone duality along full embeddings
4. The bitopological point of view



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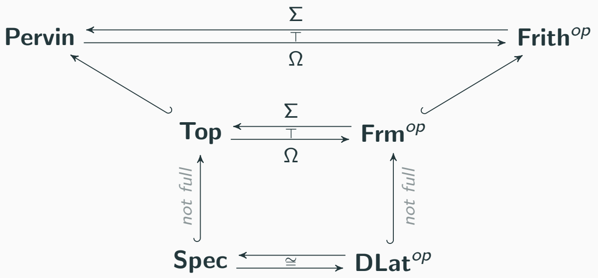

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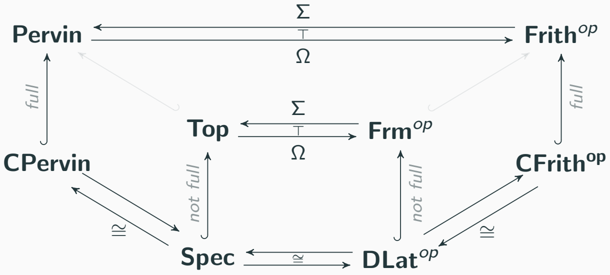

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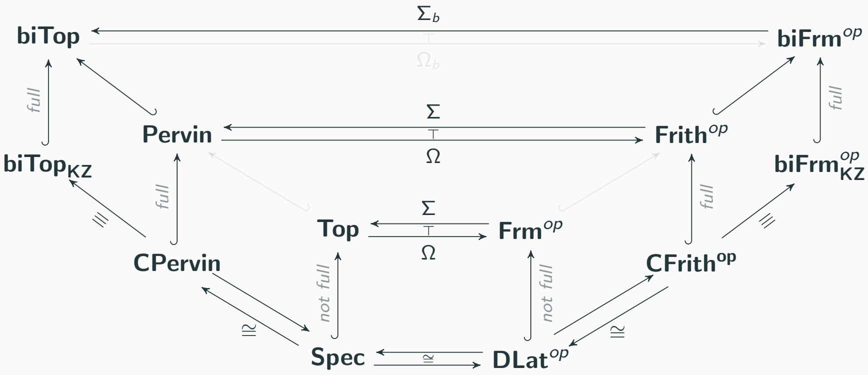

The spatial-sober duality ⏩ Skip

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Frames

A *frame* is a complete lattice L satisfying

$$a \wedge \bigvee_{i \in I} b_i = \bigvee (a \wedge b_i).$$

Frame homomorphisms preserve finite meets and arbitrary joins.

Topological spaces

and continuous functions.

The spatial-sober duality

- $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$

$$\Omega(X) := (\{\text{open subsets of } X\}, \subseteq) \qquad \Omega(X \xrightarrow{f} Y) := (\Omega(Y) \xrightarrow{f^{-1}} \Omega(X))$$

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We have an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \Sigma$ which restricts and co-restricts to a duality between sober spaces and spatial frames.

– Spatial frame: a frame of the form $\Omega(X)$ for some topological space X.

– Sober space: a space that is completely determined by its set of open subsets.

Bounded distributive lattices seen as coherent frames

A coherent frame is a frame L whose set of compact elements $K(L)$ is closed under finite meets (thus, a sublattice) and join-dense in L.

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- If D is a bounded distributive lattice, $(\mathrm{Idl}(D), \subseteq)$ is a coherent frame.
  If $h : C \to D$ is a lattice homomorphism, $\mathrm{Idl}(h) : (J \in \mathrm{Idl}(C)) \mapsto \langle h[J] \rangle_{\mathrm{Idl}}$ is a coherent homomorphism.

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- If L is a coherent frame, $K(L)$ is a bounded distributive lattice.

  If $h : L \to M$ is a coherent homomorphism, the restriction and co-restriction $K(h) : K(L) \to K(M)$ of h is a lattice homomorphism.

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These assignments define an equivalence of categories **DLat** $\cong$ **CohFrm**

Spectral spaces

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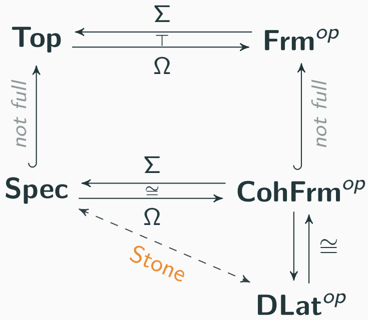
The adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \Sigma$ restricts and co-restricts to an equivalence

$$\mathbf{Spec} \cong \mathbf{CohFrm}^{op}$$

Stone duality for bounded distributive lattices



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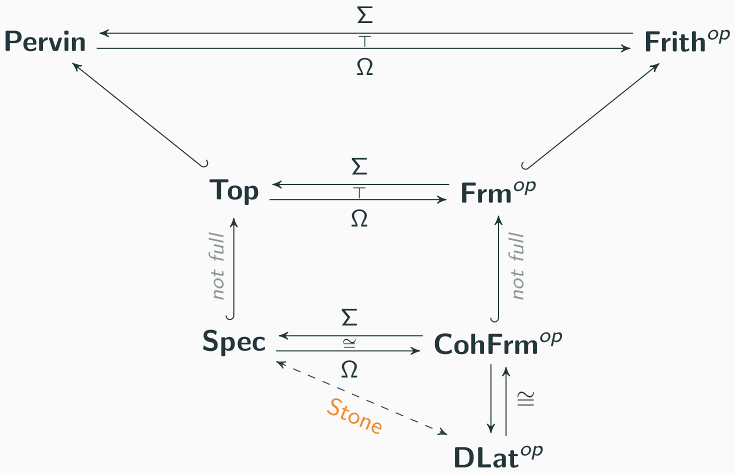

Pervin spaces

What are the quasi-uniformizable topological spaces?

Every topology comes from a quasi-uniformity!

Every topology comes from a transitive and totally bounded quasi-uniformity.

Pervin spaces

A Pervin space is a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is a bounded sublattice.

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A morphism of Pervin spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is a map $f : X \to Y$ such that for every $T \in \mathcal{T}$, we have $f^{-1}(T) \in \mathcal{S}$.

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There is a full embedding **Top** $\hookrightarrow$ **Pervin**, $\quad (X, \tau) \mapsto (X, \tau)$.

Theorem (Pin, 2017)

The category **Pervin** *of Pervin spaces is equivalent to the category of transitive and totally bounded quasi-uniform spaces.*

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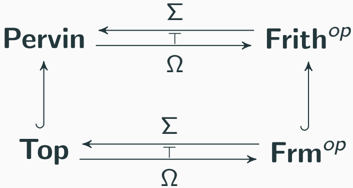
There is a full embedding $\mathbf{Frm} \hookrightarrow \mathbf{Frith}$, $\quad L \mapsto (L, L)$.

Theorem (B., Suarez)

*The category **Frith** of Frith frames is a coreflective subcategory of the category of transitive and totally bounded quasi-uniform frames.*

The Frith-Pervin adjunction



The Frith-Pervin adjunction

- $\Omega : \mathbf{Pervin} \to \mathbf{Frith}^{op}$

$$\Omega(X, \mathcal{S}) := (\langle \mathcal{S} \rangle_{\mathbf{Frm}},\ \mathcal{S})$$

$$\Omega((X, \mathcal{S}) \xrightarrow{f} (Y, \mathcal{T})) := (\Omega(Y, \mathcal{T}) \xrightarrow{f^{-1}} \Omega(X, \mathcal{S}))$$

$$\begin{array}{ccc} \mathbf{Pervin} & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{Frith}^{op} \\ \uparrow & & \uparrow \\ \mathbf{Top} & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{Frm}^{op} \end{array}$$

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- $\Sigma : \mathbf{Frith}^{op} \to \mathbf{Pervin}$

  $\Sigma(L, S) := (\Sigma(L),\ \{\widehat{s} \mid s \in S\}) \qquad (\widehat{s} := \{F \mid s \in F\})$

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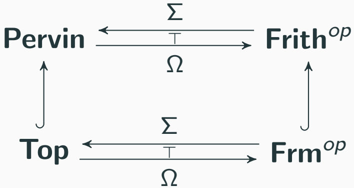
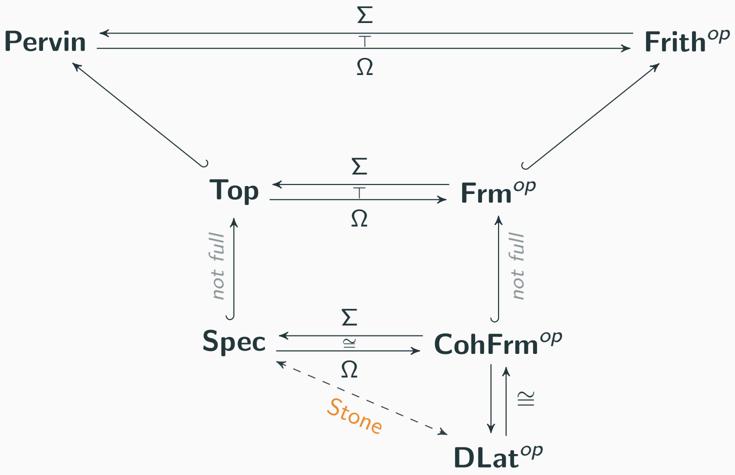


Stone duality for bounded distributive lattices



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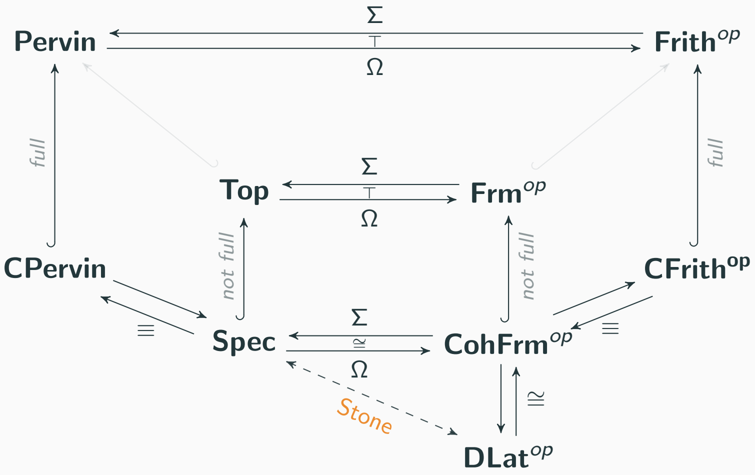

Completion of Pervin spaces

Theorem (Gehrke, Grigorieff, Pin, 2010; Pin, 2017)

The categories of spectral spaces and of complete T_0 Pervin spaces are isomorphic.

Symmetric Frith frames

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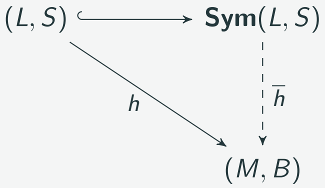
Proposition (B., Suarez)

Symmetric Frith frames form a full reflective subcategory of **Frith**.

That is, for every Frith frame (L, S), there exists a symmetric Frith frame

$$\mathbf{Sym}(L, S) = (\mathcal{C}_S L, \langle S \rangle_{\mathbf{BA}}),$$

called the symmetrization of (L, S), such that for every $h : (L, S) \to (M, B)$ with (M, B) symmetric there is a unique $\overline{h}$ making the following diagram commute:



Completion of Frith frames

- A symmetric Frith frame (L, B) is complete if every dense surjection[1] $(M, C) \twoheadrightarrow (L, B)$ with (M, C) symmetric is an isomorphism.

[1] $h : (M, C) \twoheadrightarrow (L, B)$ is a dense surjection if $(h(a) = 0 \implies a = 0)$ and $h[C] = B$.

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Corollary

The categories of coherent frames and of complete Frith frames are isomorphic.

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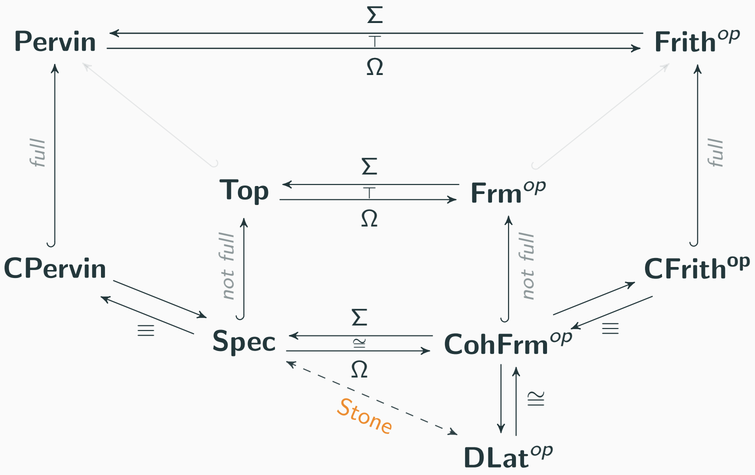

The bitopological point of view

Bitopological spaces and biframes

A bitopological space is a triple (X, τ_1, τ_2), where τ_i is a topology on X.

A biframe is a triple (L, L_1, L_2) of frames st $L_i \leq L$ and $L = \langle L_1 \cup L_2 \rangle_{\mathbf{Frm}}$.

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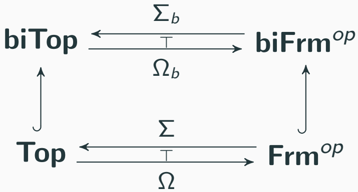
- $\Omega_b : \mathbf{biTop} \to \mathbf{biFrm}^{op}$

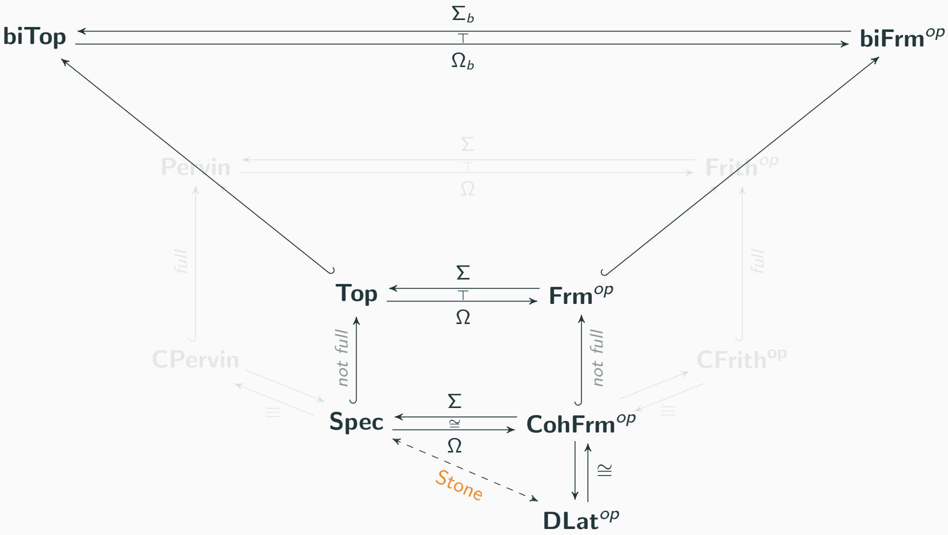
  $\Omega_b(X, \tau_1, \tau_2) := (\tau_1 \vee \tau_2,\ \tau_1,\ \tau_2)$

- $\Sigma_b : \mathbf{biFrm}^{op} \to \mathbf{biTop}$

  $\Sigma_b(L, L_1, L_2) := (\Sigma(L),\ \{\widehat{a} \mid a \in L_1\},\ \{\widehat{a} \mid a \in L_2\})$



Pairwise Stone spaces are dual to bounded distributive lattices



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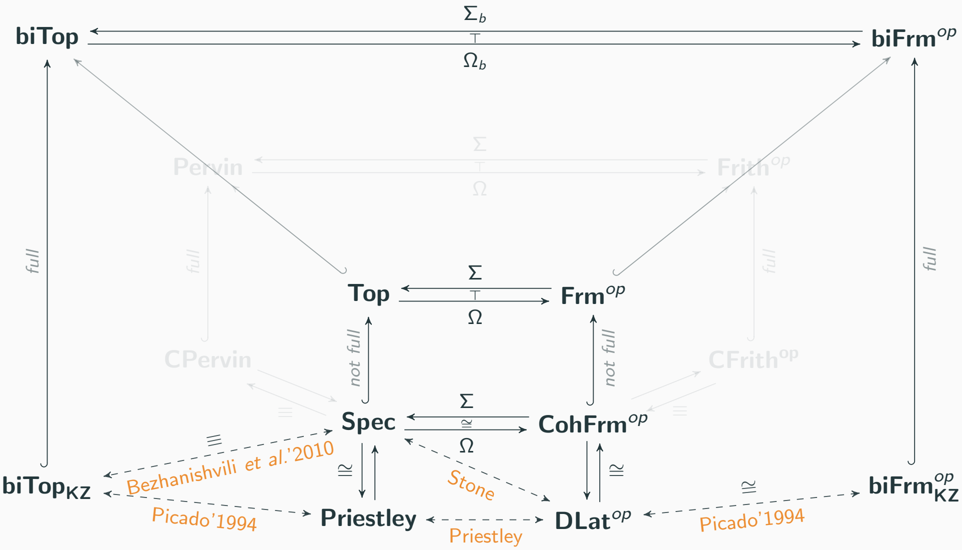

The Skula functor

If $(X, \mathcal{S})$ is a Pervin space, $(X, \langle \mathcal{S} \rangle_{\mathbf{Top}}, \langle \{U^c \mid U \in \mathcal{S}\} \rangle_{\mathbf{Top}})$ is a bispace.

This defines a full embedding

$$\mathbf{Sk}_{\text{Pervin}} : \mathbf{Pervin} \hookrightarrow \mathbf{biTop}$$

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Proposition

A bitopological space is a T_0 compact and 0-dimensional if and only if it is of the form $\mathbf{Sk}_{\mathrm{Pervin}}(X, \mathcal{S})$ for some complete T_0 Pervin space $(X, \mathcal{S})$.

The bitopological space (X, τ_1, τ_2) is:

- T_0 if $(X, \tau_1 \vee \tau_2)$ is T_0;
- compact if $(X, \tau_1 \vee \tau_2)$ is compact;
- 0-dimensional if $\{U \in \tau_k \mid U^c \in \tau_\ell\}$ is a basis for τ_k, where $\{k, \ell\} = \{1, 2\}$.

The Skula functor

If (L, S) is a Frith frame, $(\mathcal{C}_S L,\ \nabla L,\ \langle \{\Delta_s \mid s \in S\} \rangle_{\mathbf{Frm}})$ is a biframe.

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Proposition

A biframe is compact and 0-dimensional if and only if it is of the form $\mathbf{Sk}_{\mathrm{Frith}}(L, S)$ for some complete Frith frame (L, S).

The biframe (L, L_1, L_2) is:

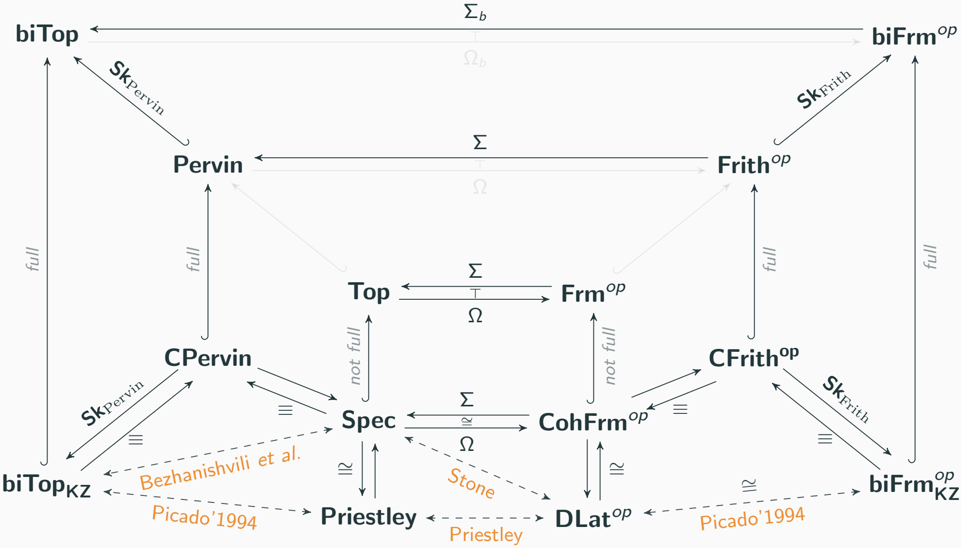
- compact if L is compact;
- 0-dimensional if $L_i = \langle\{a \in L_i \text{ complemented} \mid \neg a \in L_j\}\rangle_{\mathbf{Frm}}$ with $\{i, j\} = \{1, 2\}$.

The Skula functor

Proposition (B., Suarez)

The following square commutes up to natural isomorphism.

$$\begin{array}{ccc} \mathbf{Frith}^{op} & \xhookrightarrow{\mathbf{Sk}_{\mathrm{Frith}}} & \mathbf{biFrm}^{op} \\ \Sigma \big\downarrow & & \big\downarrow \Sigma_b \\ \mathbf{Pervin} & \xhookrightarrow[\mathbf{Sk}_{\mathrm{Pervin}}]{} & \mathbf{biTop} \end{array}$$



biTop
Σ_b
$\top$
Ω_b
$\mathbf{biFrm}^{op}$
$\mathbf{Sk}_{\mathrm{Pervin}}$
$\mathbf{Sk}_{\mathrm{Frith}}$
Pervin
Σ
$\top$
Ω
$\mathbf{Frith}^{op}$
full
full
full
full
Top
Σ
$\top$
Ω
$\mathbf{Frm}^{op}$
not full
not full
CPervin
$\mathbf{CFrith}^{op}$
$\mathbf{Sk}_{\mathrm{Pervin}}$
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$\equiv$
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Spec
Σ
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Ω
$\mathbf{CohFrm}^{op}$
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Bezhanishvili et al.
Stone
$\cong$
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Picado'1994
Priestley
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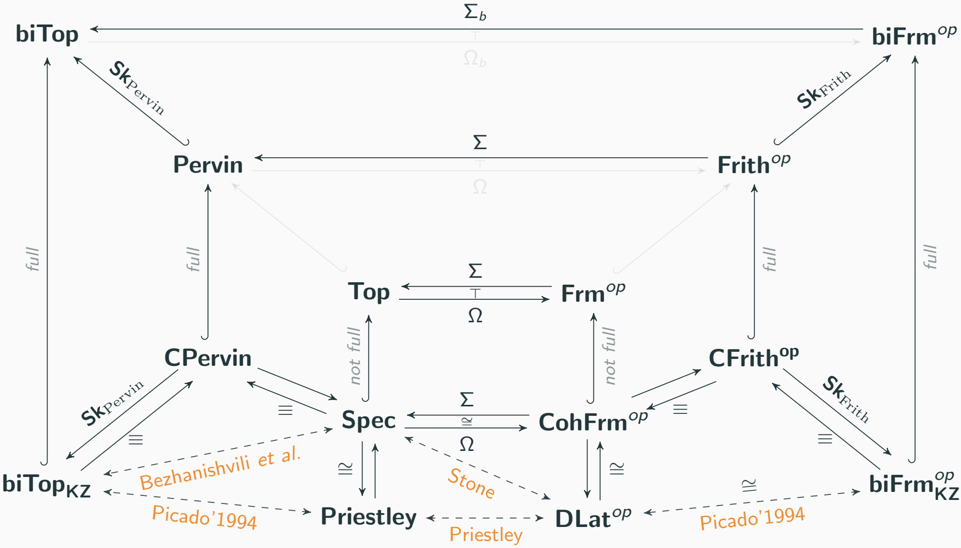

Thank you for your attention!