Measurable functions on σ-frames

A Day on Pointfree Topology

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Frame of reals and frame of extended recall

Recall:

The **frame of reals** is the frame $\mathfrak{L}(\mathbb{R})$ generated by all elements $(p, —)$ and $(—, q)$, with $p, q \in \mathbb{Q}$, and relations

(R_1) $(p, —) \wedge (—, q) = 0$ whenever $p \geq q$;

(R_2) $(p, —) \vee (—, q) = 1$ whenever $p < q$;

(R_3) $(p, —) = \bigvee\{(r, —) \mid p < r\}$;

(R_4) $(—, q) = \bigvee\{(—, s) \mid s < q\}$;

(R_5) $1 = \bigvee\{(p, —) \mid p \in \mathbb{Q}\}$;

(R_6) $1 = \bigvee\{(—, q) \mid q \in \mathbb{Q}\}$.

The **frame $\mathfrak{L}(\overline{\mathbb{R}})$ of extended reals** is the frame generated by all $(p, —)$ and $(—, q)$, with $p, q \in \mathbb{Q}$, subject to the relations (R_1)-(R_4).

Also recall that:

Proposition
Let L be a σ-frame with a countable set of generators. Then L is a frame, and $\sigma\mathsf{Frm}(L, M) = \mathsf{Frm}(L, M)$ for any frame M.

Localic real and extended real functions

From now one, we will mainly work on a σ-frame L.

Definition: A **localic real-valued function** on L is a σ-frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ and

$$\mathsf{F}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L))$$

Definition: A **localic extended real-valued function** on L is a σ-frame homomorphism $f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$, and

$$\overline{\mathsf{F}}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L))$$

- We say that an extended real function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ is **finite** if $f(\omega) = 1$, where $\omega = \left(\bigvee_{p\in\mathbb{Q}}(p, \text{—})\right) \wedge \left(\bigvee_{q\in\mathbb{Q}}(\text{—}, q)\right)$, and we have that

$$\{f \in \overline{\mathsf{F}}(L) | f \text{ is finite}\} \cong \mathsf{F}(L) \qquad (\text{Recall: } \downarrow\omega \cong \mathfrak{L}(\mathbb{R}))$$

Measurable functions

Measurable and semimeasurable funcions

Given an extended real function $f : \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ on L:

Definitions:

1. We say that f is **lower measurable** (resp. **upper measurable**) if $f(r, —) \in \nabla[L]$ for every $r \in \mathbb{Q}$ (resp. $f(—, r) \in \nabla[L]$) for every $r \in \mathbb{Q}$), and we denote by $\overline{\mathsf{LM}}(L)$ and $\overline{\mathsf{UM}}(L)$ the corresponding collections of lower measurable and upper measurable extended real functions.

2. Whenever $f \in \overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L)$, we say that f is **measurable**, and we shall denote $\overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L)$ by $\overline{\mathsf{M}}(L)$. In other words, f is measurable if

$$f(p, q) \in \nabla[L], \forall p, q \in \mathbb{Q}$$

Remarks

- $\overline{\mathsf{M}}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L)$ as $\nabla[L] \cong L$;
- $\overline{\mathsf{M}}(\mathcal{C}(L)) = \overline{\mathsf{F}}(L)$;

- A measurable function $f : \mathfrak{L}(\overline{\mathbb{R}}) \to L$ preserves all joins despite the fact that L has not necessarily arbitrary joins, that is, for any $A \subseteq \mathfrak{L}(\overline{\mathbb{R}})$,

$$\bigvee_{a\in A} f(a) \text{ exists in } L \text{ and } \bigvee_{a\in A} f(a) = f\Big(\bigvee_{a\in A} a\Big).$$

Finite valued case

Restricting to the finite-valued case, we introduce the classes

$$\mathsf{LM}(L) := \overline{\mathsf{LM}}(L) \cap \mathsf{F}(L) \quad \text{(of lower measurable real functions);}$$
$$\mathsf{UM}(L) := \overline{\mathsf{UM}}(L) \cap \mathsf{F}(L) \quad \text{(of upper measurable real functions);}$$
$$\mathsf{M}(L) := \overline{\mathsf{M}}(L) \cap \mathsf{F}(L) \quad \text{(of measurable real functions).}$$

We have

$$\begin{array}{ccc} \overline{\mathsf{M}}(L) & \subseteq & \overline{\mathsf{F}}(L) \\ \cup\!\!| & & \cup\!\!| \\ \mathsf{M}(L) & \subseteq & \mathsf{F}(L). \end{array}$$

σ-scales

Aside

In a frame L we have:

extended scales (maps $\sigma\colon \mathbb{Q} \to L$ s.t. $\sigma(r) \prec \sigma(s)$ whenever $r < s$) generating continuous extended real functions, and

scales (extended scales $\sigma\colon \mathbb{Q} \to L$ s.t. $\bigvee_{r\in\mathbb{Q}} \sigma(r) = 1 = \bigvee_{r\in\mathbb{Q}} \sigma(r)^*$) generating continuous real functions.

Recall:

$$a \prec b \equiv a^* \vee b = 1$$
$$\equiv \exists u \in L : a \wedge u = 0 \text{ and } u \vee b = 1.$$
$$a \prec\!\!\prec b \equiv \exists a_q \in L, q \in [0,1] \cap \mathbb{Q} : a_0 = a, a_1 = b \text{ and } a_p \prec a_q \; (p < q).$$

σ-scales

Definition: A map $\varphi\colon \mathbb{Q} \to L$ is a **σ-scale** in L (or an **ascending σ-scale**) if there exists a family $(c_r)_{r\in\mathbb{Q}}$ of elements of L such that

$$\varphi(s) \wedge c_r = 0 \text{ whenever } s \leq r \text{ and}$$
$$c_r \vee \varphi(s) = 1 \text{ whenever } r < s.$$

Furthermore, we say that φ is **finite** if $\bigvee_{r\in\mathbb{Q}} \varphi(r) = 1 = \bigvee_{r\in\mathbb{Q}} c_r$.

σ-scales

Proposition
Given a map $\varphi\colon \mathbb{Q} \to L$*:*

$$\varphi \text{ is a } \sigma\text{-scale } \textbf{iff } \varphi(r) \prec \varphi(s) \text{ whenever } r < s.$$

φ is a finite σ-scale **iff** *φ is a σ-scale such that $\bigvee_{r\in\mathbb{Q}} \varphi(r) = 1$ and there are $c_{rs} \in L$ such that $\bigvee\{c_{rs} \mid r, s \in \mathbb{Q}, r < s\} = 1$, with $\varphi(r) \wedge c_{rs} = 0$ and $c_{rs} \vee \varphi(s) = 1$ whenever $r < s$.*

CONSEQUENTLY: If L is a frame, then

(1) σ-scales in L are precisely the extended scales in L.

(2) finite σ-scales in L are precisely the scales in L.

σ-scales

Proposition: Let L be a σ-frame. Given a σ-scale $\varphi\colon \mathbb{Q} \to L$ and a family $(c_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$, the map $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ determined by

$$f(p,\text{—}) = \bigvee_{r>p} c_r \text{ and } f(\text{—},q) = \bigvee_{r<q} \varphi(r) \quad (p,q \in \mathbb{Q})$$

is a measurable function on L. Moreover, if φ is finite, then f is a finite-valued function.

REMARK: As $\overline{\mathsf{F}}(L) = \overline{\mathsf{M}}(\mathcal{C}(L))$ and $\mathsf{F}(L) = \mathsf{M}(\mathcal{C}(L))$, σ-scales and finite σ-scales in $\mathcal{C}(L)$ generate extended real and real-valued functions on L.

Insertion, extension and separation results

Katětov relation

Definition: A **Katětov relation** is a binary relation $\Subset$ on a lattice L satisfying the following conditions for all $a, b, a', b' \in L$:

(K_1) $a \Subset b \Rightarrow a \leq b$;
(K_2) $a' \leq a, a \Subset b, b \leq b' \Rightarrow a' \Subset b'$;
(K_3) $a \Subset b, a' \Subset b \Rightarrow (a \vee a') \Subset b$;
(K_4) $a \Subset b, a \Subset b' \Rightarrow a \Subset (b \wedge b')$;
(K_5) $a \Subset b \Rightarrow \exists c \in L : a \Subset c \Subset b$.

Katětov Lemma: Let $\Subset$ be a Katětov relation on L and $\lhd$ a transitive and irreflexive (i.e, a relation that is not reflexive) relation on a countable set D. Consider two families $(a_d)_{d \in D}$ and $(b_d)_{d \in D}$ of elements of L such that

$$d_1 \lhd d_2 \text{ implies } a_{d_2} \leq a_{d_1}, b_{d_2} \leq b_{d_1} \text{ and } a_{d_2} \Subset b_{d_1}.$$

Then there exists a family $(c_d)_{d \in D}$ in L such that

$$d_1 \lhd d_2 \text{ implies } c_{d_2} \Subset c_{d_1}, a_{d_2} \Subset c_{d_1} \text{ and } c_{d_2} \Subset b_{d_1}.$$

The relations $\Subset_{\overline{M}}$ and $\Subset_M$

For any $\theta_A, \theta_B \in \mathcal{C}(L)$, define

$$\theta_A \Subset_{\overline{M}} \theta_B \equiv \exists f \in \overline{\mathsf{M}}(L) \colon \theta_A \subseteq f(p,-)^* \text{ and } f(-,q) \subseteq \theta_B \text{ for some } p < q.$$

We write $\theta_A \Subset_M \theta_B$ whenever $f \in \mathsf{M}(L)$.

Lemma: For any $\theta_A, \theta_B \in \mathcal{C}(L)$ we have:

(1) $\theta_A \Subset_{\overline{M}} \theta_B$ if and only if there is some $f \in \overline{\mathsf{M}}(L)$ such that $\theta_A \subseteq f(0,-)^*$ and $f(-,1) \subseteq \theta_B$. Moreover, $\theta_A \Subset_M \theta_B$ if and only if such f is finite-valued.

(2) If $\theta_A \Subset_{\overline{M}} \theta_B$ then $\theta_B^* \Subset_{\overline{M}} \theta_A^*$. In particular, if $\theta_A \Subset_M \theta_B$ then $\theta_B^* \Subset_M \theta_A^*$.

Proposition: Both $\Subset_{\overline{M}}$ and $\Subset_M$ are Katětov relations on $\mathcal{C}(L)$.

Separating relations on $\mathcal{C}(L)$

Given a σ-frame L, a relation $R \subseteq \mathcal{C}(L)$ is **separating** if $\theta_A R\, \theta_B$ implies the existence of $a, b \in L$ such that $\theta_A \subseteq \Delta_a \subseteq \theta_B$ and $\theta_A \subseteq \nabla_b \subseteq \theta_B$.

Proposition: Let L be a σ-frame, φ a σ-scale in $\mathcal{C}(L)$ and R a separating relation on $\mathcal{C}(L)$ such that $\varphi(r) R\, \varphi(s)$ whenever $r < s$. Then the function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ generated by φ is measurable. In particular, if φ is finite, then f is finite-valued.

Basic Insertion Theorem

Basic Insertion Theorem: Given functions $g, h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ on a σ-frame L such that $g \leq h$, the following statements are equivalent:

(i) There exists a measurable function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $g \leq f \leq h$.

(ii) For each $p < q$, $h(p, —)^* \Subset_{\overline{M}} g(—, q)$.

(iii) There exist σ-scales φ_1 and φ_2 generating g and h, respectively, such that $\varphi_2(r) \Subset_{\overline{M}} \varphi_1(s)$ whenever $r < s$.

(iv) There exist σ-scales φ_1 and φ_2 generating g and h, respectively, and a separating Katětov relation R on $\mathcal{C}(L)$ such that $\varphi_2(r) R\, \varphi_1(s)$ whenever $r < s$.

Corollary: Let θ_A, θ_B be complemented congruences on a σ-frame L such that $\theta_A \subseteq \theta_B$. There exists a measurable function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\chi_{\theta_A} \leq f \leq \chi_{\theta_B}$ if and only if $\theta_B^* \Subset_M \theta_A^*$.

An extension condition

Consider a σ-sublocale S of L and the σ-frame homomorphism $q_S\colon \mathcal{C}(L) \to \mathcal{C}(S)$ given by $q_S(\theta) = \theta \vee \theta_S$.

Definition: Let $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(S)$ be a function on S. A function $\widetilde{f}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ is an **extension of f over L** if $f = q_S \circ \widetilde{f}$.

Proposition: Let S be a complemented σ-sublocale of a σ-locale L and let $f\colon \mathfrak{L}(\mathbb{R}) \to S$ be a measurable function such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. The following statements are equivalent:

(i) f has a finite-valued measurable extension over L.

(ii) For each $p < q$, $f(p, —)^{*_L} \Subset_M f(—, q)$ in $\mathcal{C}(L)$.

A separation condition

Proposition: Given a σ-frame L and closed congruences $\nabla_a \subseteq \nabla_b$, there exists a measurable $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\chi_{\nabla_a} \leq f \leq \chi_{\nabla_b}$ if and only if $a \prec\!\!\prec b$.

Proposition: Given a σ-frame L and open congruences $\Delta_a \subseteq \Delta_b$, there exists a measurable $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\chi_{\Delta_a} \leq f \leq \chi_{\Delta_b}$ if and only if $b \prec\!\!\prec a$.

For normal and extremally disconnected σ-frames

Consider the relations $\Subset_N$ and $\Subset_D$ on $\mathcal{C}(L)$ given by

$$\theta_A \Subset_N \theta_B \;\equiv\; \exists\, u, v \in L : \theta_A \subseteq \Delta_u \subseteq \nabla_v \subseteq \theta_B$$
$$\text{and } \theta_A \Subset_D \theta_B \;\equiv\; \exists\, u, v \in L : \theta_A \subseteq \nabla_u \subseteq \Delta_v \subseteq \theta_B.$$

REMARK:

$$L \text{ normal } \Rightarrow \Subset_N \text{ is a is a separating Katětov relation}$$
$$L \text{ extremally disconnected } \Rightarrow \Subset_D \text{ is a is a separating Katětov relation}$$

Lemma: For any σ-frame L, $\Subset_M \subseteq \Subset_N \cap \Subset_D$.

Characterisation of normality

Theorem The following statements are equivalent for a σ-frame L.

(i) L is normal, i.e., for all $a, b \in L$,

$$a \vee b = 1 \Rightarrow \exists u, v \in L : u \wedge v = 0 \text{ and } a \vee u = 1 = b \vee v.$$

(ii) (Insertion) For any $g \in \overline{\mathsf{UM}}(L)$ and $h \in \overline{\mathsf{LM}}(L)$ such that $g \leq h$, there exists an $f \in \overline{\mathsf{M}}(L)$ such that $g \leq f \leq h$.

(iii) (Insertion) For any $g \in \mathsf{UM}(L)$ and $h \in \mathsf{LM}(L)$ such that $g \leq h$, there exists an $f \in \mathsf{M}(L)$ such that $g \leq f \leq h$.

(iv) (Separation) For every $a, b \in L$, $a \vee b = 1$ implies that $\Delta_a \Subset_M \nabla_b$.

(v) (Extension) For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has a finite-valued measurable extension over L.

Characterisation of extremally disconnectedness

Theorem The following statements are equivalent for a σ-frame L.

(i) L is extremally disconnected, i.e., for all $a, b \in L$

$$a \wedge b = 0 \Rightarrow \exists u, v \in L : u \vee v = 1 \text{ and } a \wedge u = 0 = b \wedge v.$$

(ii) (Insertion) For any $g \in \overline{\mathsf{LM}}(L)$ and $h \in \overline{\mathsf{UM}}(L)$ such that $g \leq h$, there exists an $f \in \overline{\mathsf{M}}(L)$ such that $g \leq f \leq h$.

(iii) (Insertion) For any $g \in \mathsf{LM}(L)$ and $h \in \mathsf{UM}(L)$ such that $g \leq h$, there exists an $f \in \mathsf{M}(L)$ such that $g \leq f \leq h$.

(iv) (Separation) For every $a, b \in L$, $a \wedge b = 0$ implies that $\nabla_a \Subset_M \Delta_b$.

(v) (Extension) For each open σ-sublocale S of L, every $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has a finite-valued measurable extension over L.

Characterisation of $\mathcal{F}$-perfectness

Theorem: The following statements are equivalent for a σ-frame L.

(i) L is $\mathcal{F}$-perfect, i.e., for each $a \in L$ there is a sequence $(a_i)_{i\in\mathbb{N}} \subseteq L$ such that $\Delta_a = \bigwedge_{i\in\mathbb{N}} \nabla_{a_i}$.

(ii) (Insertion) For any $-u, l \in \overline{\mathsf{LM}}(L)$ such that l and $-u$ are sum compatible and $\mathbf{0} \leq l - u$, there exist $u_1 \in \mathsf{UM}(L)$ and $l_1 \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u_1 \leq l - u$, $u - l + u_1 \leq l_1 \leq \mathbf{0}$ and

$$(l - u)(0, \text{—})^* = u_1(0, \text{—})^* = (-l_1)(0, \text{—})^*.$$

(iii) (Insertion) For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \leq l$, there exist $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $u \leq u' \leq l' \leq l$ and

$$(u' - u)(0, \text{—})^* = (l - l')(0, \text{—})^* = (l - u)(0, \text{—})^*.$$

Characterisation of $\mathcal{F}$-perfectness

(iv) (Extension) For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has an upper measurable extension $u' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ and a lower measurable extension $l' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$ and

$$\theta_S^* \vee u'(0,\text{—})^* = \theta_S^* \vee l'(\text{—},1)^* = \theta_S^*.$$

(v) (Separation) For any $a, b \in L$ such that $a \vee b = 1$, there are $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$,

$$\begin{aligned} & u'(0,\text{—})^* \vee l'(\text{—},1)^* \subseteq \Delta_{a\wedge b}, \\ & \Delta_a \subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \text{ for all } p<0, q>0, \\ \text{and } \quad & \Delta_b \subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \text{ for all } p<1, q>1. \end{aligned}$$

Characterisation of $\mathcal{G}$-perfectness

Theorem: The following statements are equivalent for a σ-frame L.

(i) L is $\mathcal{G}$-perfect, i.e., for each $a \in L$ there is a sequence $(a_i)_{i\in\mathbb{N}} \subseteq L$ such that $\nabla_a = \bigvee_{i\in\mathbb{N}} \Delta_{a_i}$.

(ii) (Insertion) For any $-u, l \in \overline{\mathsf{LM}}(L)$ such that l and $-u$ are sum compatible and $\mathbf{0} \leq l - u$, there exist $u_1 \in \mathsf{UM}(L)$ and $l_1 \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u_1 \leq l - u$, $u - l + u_1 \leq l_1 \leq \mathbf{0}$ and

$$(l-u)(0,\text{—}) = u_1(0,\text{—}) = (-l_1)(0,\text{—}).$$

(iii) (Insertion) For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \leq l$, there exist $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $u \leq u' \leq l' \leq l$ and

$$(u'-u)(0,\text{—}) = (l-l')(0,\text{—}) = (l-u)(0,\text{—}).$$

Characterisation of $\mathcal{G}$-perfectness

(iv) (Extension) For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has an upper measurable extension $u' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ and a lower measurable extension $l' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$ and

$$\theta_S \wedge u'(0,\text{—}) = \theta_S \wedge l'(\text{—},1) = \theta_S.$$

(v) (Separation) For any $a, b \in L$ such that $a \vee b = 1$, there are $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$,

$$\begin{aligned} \nabla_{a\wedge b} &\subseteq u'(0,\text{—}) \wedge l'(\text{—},1), \\ \Delta_a &\subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \text{ for all } p<0, q>0, \\ \text{and} \quad \Delta_b &\subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \text{ for all } p<1, q>1. \end{aligned}$$

Characterisation of perfect normality

Theorem: The following statements are equivalent for a σ-frame L.

(i) L is perfectly normal, i.e., normal and $\mathcal{F}$-perfect ($\equiv \mathcal{G}$-perfect).

(ii) L is regular, i.e., $\forall a \in L, a = \bigvee_{n\in\mathbb{N}} a_n$, with $a_n \prec a$.

(iii) (Insertion) For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \leq l$, there exists an $f \in \mathsf{M}(L)$ such that $u \leq f \leq l$ and

$$(f-u)(0,\text{—}) = (l-f)(0,\text{—}) = (l-u)(0,\text{—}).$$

(iv) (Extension) For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has a measurable extension $\widetilde{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that

$$\theta_S \subseteq \widetilde{f}(0,\text{—}) \wedge \widetilde{f}(\text{—},1).$$

(v) (Separation) For every $a, b \in L$ such that $a \vee b = 1$, there exists an $f \in \mathsf{M}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$,

$$\Delta_b \subseteq f(p,\text{—}) \wedge f(\text{—},q) \text{ for all } p<1, q>1,$$

$$\Delta_a \subseteq f(p,\text{—}) \wedge f(\text{—},q) \text{ for all } p<0, q>0,$$

$$\text{and } \nabla_{a\wedge b} \subseteq f(0,\text{—}) \wedge f(\text{—},1).$$

Summing up

$\mathcal{F}$-perfectness	$+$	Normality	$=$	Perfect normality
insertion		insertion		insertion
separation		insertion		separation
extension		insertion		extension

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