Revisiting localic T_1-type separation

A day on Pointfree Topology
Celebrating Jorge Picado's 60th birthday

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A T_0-SPACE X IS T_1 IFF	ANALOGY FOR A LOCALE L
Every open subspace is a union of closed subspaces	Every open sublocale is a join of closed sublocales **Subfit**
Every (closed) subspace is saturated	Every (closed) sublocale is fitted **Fit**
Every point is closed	Every one-point sublocale is closed **T_1-locale**
Every point is saturated	Every one-point sublocale is fitted **pt-fit**
For any space Y and any continuous $f, g\colon Y \to X$, $f \leq g \implies f = g$	For any frame M and any $f, g\colon L \to M$ in Frm, $f \leq g \implies f = g$ **Totally unordered**
The diagonal is saturated in $X \times X$	The diagonal is fitted in $L \oplus L$ **$\mathcal{F}$-separated**

Subfitness:

$$a \not\leq b \implies \exists c \in L: \; a \vee c = 1 \neq b \vee c$$

Very useful property in pointfree topology. A few examples:

- Subfit + normal $\implies$ completely regular.
- Under subfitness, a frame homomorphism is open iff it has a left adjoint.
- Under subfitness, one-to-one frame homomorphisms are exactly the codense ones ($h(a) = 1 \implies a = 1$).
- Closed surjections are not always regular epimorphisms in Loc. However, under subfitness, they are.
- A frame quasi-admits a nearness iff it is subfit.

However, it is not closed under products nor sublocales.

Fitness:

$$a \not\leq b \implies \exists c \in L: \; a \vee c = 1, \; c \to b \neq b$$

The hereditary variant of subfitness. Actually fit locales are closed under all limits in Loc. Already quite close to regularity.

T_1-locales and pt-fit locales: Too point dependent; however useful as a weak separation property in the study of (non-)spatiality. Implied by several genuinely pointfree axioms.

Totally unordered locales (T_U): A natural property, well-behaved categorically, but not well understood. It is implied by both Hausdorff-type axioms and T_1-type axioms.

$\mathcal{F}$-separatedness: The natural closure-theoretic T_1-type separation. Excellent categorical properties. It is in a pleasant parallel with the strong Hausdorff property.

Other properties:

- Weak subfitness: $a \neq 0 \implies \exists c \neq 1 : \; a \vee c = 1.$
- Prefitness: $a \neq 0 \implies \exists c \neq 0 : \; a \vee c^* = 1.$

Closed hereditary prefitness = fitness;

Closed hereditary weak subfitness = subfitness.

$\mathcal{F}$-separation and strong Hausdorff

Let $\mathcal{C}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ and a closure operator c. An object is **c-separated** if the diagonal $\Delta_X \colon X \to X \times X$ is c-closed.

If $\mathcal{C} = \mathrm{Loc}$, we consider the following two closure operators:

- If $c =$ **usual closure**

$$S \mapsto \overline{S} = \bigcap_{S \subseteq \mathfrak{c}(a)} \mathfrak{c}(a).$$

  Then c-separated objects = strongly Hausdorff locales (i.e. locales with closed diagonal).
- If $c =$ **fitting operator**

$$S \mapsto S^{\circ} = \bigcap_{S \subseteq \mathfrak{o}(a)} \mathfrak{o}(a).$$

  Then c-separated objects = $\mathcal{F}$-separated locales (i.e. locales with fitted diagonal).

Because of general categorical results, we have the following:

Proposition

Strongly Hausdorff (resp. $\mathcal{F}$-separated locales) are closed under mono-sources in Loc. *In particular,*

- *They are closed under limits in* Loc*;*
- *If $M \to L$ is a monomorphism in* Loc *and L is strongly Haudorff (resp. $\mathcal{F}$-separated), then so is M.*

In Loc, the structure of monorphisms is fairly wild; and so this property is somewhat stronger than heredity under sublocales (=regular monomorphisms)!

Examples of the strongly Hausdorff-$\mathcal{F}$-separated duality: Dowker-Strauss characterizations

Strong Hausdorff

Let $h, k\colon L \to M$ be frame homomorphisms. We say that (h, k) **respect disjoint pairs** if whenever $D = \{a, b\}$ with $\bigwedge D = 0$, then

$$\bigwedge_{x \in D} h(x) \vee k(x) = 0.$$

Theorem (Dowker-Strauss)

A frame L is strongly Hausdorff if and only if no distinct frame homomorphisms $h, k\colon L \to M$ respect disjoint pairs.

$\mathcal{F}$-separated

Let $h, k\colon L \to M$ be frame homomorphisms. We say that (h, k) **respect covers** if whenever $C \subseteq L$ with $\bigvee C = 1$, then

$$\bigvee_{x \in C} h(x) \wedge k(x) = 1.$$

Theorem

A frame L is $\mathcal{F}$-separated if and only if no distinct frame homomorphisms $h, k\colon L \to M$ respect covers.

Examples of the strongly Hausdorff-$\mathcal{F}$-separated duality: relaxed morphisms

Strong Hausdorff

A map $h\colon L \to M$ between frames is a **weak homomorphism** if

1. it is a morphism in Sup,
2. $h(1) = 1$, and
3. it preserves disjoint pairs

A frame L has property **(W)** if every weak homomorphism $h\colon L \to M$ is a frame homomorphism.

Theorem (Banaschewski, Pultr)

Strong Hausdorff $\equiv (T_U)$ + (W).

$\mathcal{F}$-separated

A map $h\colon L \to M$ between frames is an **almost homomorphism** if

1. it is a morphism in PreFrm,
2. $h(0) = 0$, and
3. it preserves covers

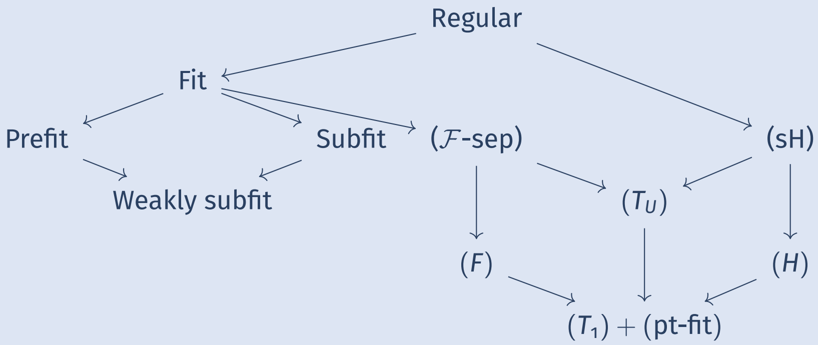
A frame L has property **(A)** if every almost homomorphism $h\colon L \to M$ is a frame homomorphism.

Theorem

$\mathcal{F}$-separated $\equiv (T_U)$ + (A).

STRONG HAUSDORFF L	$\mathcal{F}$-SEPARATED L
No distinct pair of frame homomorphisms $f, g\colon L \to M$ preserves disjoint pairs	No distinct pair of frame homomorphisms $f, g\colon L \to M$ preserves covers
Strong Hausdorff implies (T_U)	$\mathcal{F}$-separated implies (T_U)
Strongly Hausdorff = $(T_U) + (W)$	$\mathcal{F}$-separated = $(T_U) + (A)$
Hereditary normality $\implies$ (W)	Hereditary extremal disconnectedness $\implies$ (A)
Dwn(X) satisfies (W) iff it is hereditarily normal	Dwn(X) satisfies (A) iff it is hereditarily extremally disconnected
It implies the conservative Hausdorff axiom (H): If $1 \neq a \not\leq b$, then $\exists u, v \in L$: $u \wedge v = 0,\ u \not\leq a,\ v \not\leq b$	It implies a certain new property (F): If $1 \neq a \not\leq b$, then $\exists u, v \in L$: $u \vee v = 1,\ u \to a \not\leq a,\ v \to b \not\leq b$

Reviewing localic T_1 and T_2 separation



The only implications that hold are the ones that follow from the diagram!

Categorical compactness

Let $\mathcal{C}$ be a finitely complete category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ and c a closure operator in $\mathcal{C}$. An object X of $\mathcal{C}$ is ***c*-compact** if the projection

$$\pi_2 : X \times Y \longrightarrow Y$$

is c-closed for any object Y.

- In Top, c=usual Kuratowski closure, c-compact spaces=compact spaces.
  In Loc, c= usual closure, c-compact locales=compact locales.
- In Top, c = saturation closure, then any topological space is c-compact! But its proof uses the fact that the subobject lattice is completely distributive...

If $c = \mathcal{F}$ is the fitting closure operator, what are the $\mathcal{F}$-compact objects in Loc?

The situation in Loc is very different from that in Top:

Proposition

If X is a Hausdorff topological space such that $\Omega(X)$ is fit, then $\Omega(X)$ is $\mathcal{F}$-compact if and only if X is finite.

Hence, no infinite regular space can be $\mathcal{F}$-compact.

However, there is still an interesting class of spaces (locales) that are $\mathcal{F}$-compact.

A. H. Stone
Hereditarily compact spaces
American Journal of Mathematics 82 (1960), 900-916 .

A space is **semi-irreducible** if every pairwise-disjoint family of nonempty open sets is finite.

Irreducible spaces and hereditarily compact spaces are semi-irreducible.

For locales the analogous property was introduced in

T. Dube
Irreducibility in pointfree topology
Quaestiones Mathematicae 27 (2004), 231-241 .

We have the following positive result.

Proposition

Every semi-irreducible locale is $\mathcal{F}$-compact.

Conjecture: A locale is semi-irreducible iff it is $\mathcal{F}$-compact.

Equivalent conjecture: A Boolean locale which is $\mathcal{F}$-compact must be finite (cf. the fact that a Boolean locale which is compact must be finite).

Happy birthday, Jorge!