The geometry of locale maps via Galois adjunctions

João Areias

12 October 2023

Motivation

Given two partially ordered sets X and Y, a Galois adjunction between them consists of a pair of order-preserving maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$f(x) \leq y \iff x \leq g(y)$$

for all $x \in X$ and $y \in Y$.

Motivation

Given two partially ordered sets X and Y, a Galois adjunction between them consists of a pair of order-preserving maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$f(x) \leq y \iff x \leq g(y)$$

for all $x \in X$ and $y \in Y$. One calls f *left adjoint* to g and g *right adjoint* to f and writes $f \dashv g$.

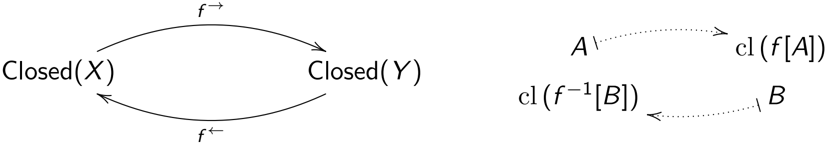
Motivation

Let $f\colon X \to Y$ be a function between topological spaces and define the pair of assignments

$$\mathrm{Closed}(X) \underset{f^{\leftarrow}}{\overset{f^{\rightarrow}}{\rightleftarrows}} \mathrm{Closed}(Y) \qquad A \mapsto \mathrm{cl}\,(f[A]) \qquad \mathrm{cl}\,(f^{-1}[B]) \mapsfrom B$$

Motivation

Let $f\colon X \to Y$ be a function between topological spaces and define the pair of assignments



Then it is very easy to check that

f is continuous iff $(f^{\rightarrow}, f^{\leftarrow})$ is an adjoint pair.

Closure operator

For any subset $S \subseteq L$, let $\operatorname{cl} S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

Closure operator

For any subset $S \subseteq L$, let $\mathrm{cl}\, S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

This defines a map from $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the following properties:

Closure operator

For any subset $S \subseteq L$, let $\mathrm{cl}\, S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

This defines a map from $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \mathrm{cl}\, S \subseteq \mathrm{cl}\, T$.

Closure operator

For any subset $S \subseteq L$, let $\operatorname{cl} S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

This defines a map from $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \operatorname{cl} S \subseteq \operatorname{cl} T$.
- idempotent: $\operatorname{cl}(\operatorname{cl} S) = \operatorname{cl} S$.

Closure operator

For any subset $S \subseteq L$, let $\mathrm{cl}\, S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

This defines a map from $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \mathrm{cl}\, S \subseteq \mathrm{cl}\, T$.
- idempotent: $\mathrm{cl}\,(\mathrm{cl}\, S) = \mathrm{cl}\, S$.
- extensive: $S \subseteq \mathrm{cl}\, S$.

Closure operator

For any subset $S \subseteq L$, let $\mathrm{cl}\, S$ denote the closed sublocale

$$\bigcap\{\mathfrak{c}(a) \in \mathfrak{c}L \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L \mid S \subseteq \mathfrak{c}(a)\}) = \mathfrak{c}(\bigwedge S).$$

This defines a map from $\mathcal{P}(L)$ to $\mathfrak{c}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \mathrm{cl}\, S \subseteq \mathrm{cl}\, T$.
- idempotent: $\mathrm{cl}\,(\mathrm{cl}\, S) = \mathrm{cl}\, S$.
- extensive: $S \subseteq \mathrm{cl}\, S$.

Clearly, one has the equivalence

$$S \subseteq \mathrm{cl}\, T \Leftrightarrow \mathrm{cl}\, S \subseteq \mathrm{cl}\, T \text{ for every } T, S \subseteq L.$$

Adjoint pair I

For each plain map $f\colon L \to M$ between locales consider the following:

$$\mathfrak{c}L \underset{f_{\mathfrak{c}}^{\leftarrow}}{\overset{f_{\mathfrak{c}}^{\rightarrow}}{\rightleftarrows}} \mathfrak{c}M \qquad (\perp)$$

$$\mathfrak{c}(a) \mapsto \mathrm{cl}\,(f[\mathfrak{c}(a)])$$

$$\mathrm{cl}\,(f^{-1}[\mathfrak{c}(b)]) \mapsfrom \mathfrak{c}(b)$$

Adjoint pair I

Adjoint pair I

Proposition

Let $f : L \to M$ be a plain map between locales. The pair $(f_{\mathfrak{c}}^{\to}, f_{\mathfrak{c}}^{\leftarrow})$ is an adjoint pair if and only if f preserves arbitrary meets.

Interior operator

For any subset $S \subseteq L$, let $\mathrm{int}\, S$ denote the open sublocale

$$\bigvee\{\mathfrak{o}(a) \in \mathfrak{o}L \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}(a) \subseteq S\}).$$

Interior operator

For any subset $S \subseteq L$, let $\operatorname{int} S$ denote the open sublocale

$$\bigvee\{\mathfrak{o}(a) \in \mathfrak{o}L \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}(a) \subseteq S\}).$$

This defines a mapping from $\mathcal{P}(L)$ to $\mathfrak{o}L$ with the following properties:

Interior operator

For any subset $S \subseteq L$, let $\operatorname{int} S$ denote the open sublocale

$$\bigvee\{\mathfrak{o}(a) \in \mathfrak{o}L \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}(a) \subseteq S\}).$$

This defines a mapping from $\mathcal{P}(L)$ to $\mathfrak{o}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \operatorname{int} S \subseteq \operatorname{int} T$.

Interior operator

For any subset $S \subseteq L$, let $\mathrm{int}\, S$ denote the open sublocale

$$\bigvee\{\mathfrak{o}(a) \in \mathfrak{o}L \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}(a) \subseteq S\}).$$

This defines a mapping from $\mathcal{P}(L)$ to $\mathfrak{o}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \mathrm{int}\, S \subseteq \mathrm{int}\, T$.
- idempotent: $\mathrm{int}\,(\mathrm{int}\, S) = \mathrm{int}\, S$.

Interior operator

For any subset $S \subseteq L$, let $\operatorname{int} S$ denote the open sublocale

$$\bigvee\{\mathfrak{o}(a) \in \mathfrak{o}L \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}(\bigvee\{a \in L \mid \mathfrak{o}(a) \subseteq S\}).$$

This defines a mapping from $\mathcal{P}(L)$ to $\mathfrak{o}L$ with the following properties:

- order-preserving: $S \subseteq T \;\Rightarrow\; \operatorname{int} S \subseteq \operatorname{int} T$.
- idempotent: $\operatorname{int}(\operatorname{int} S) = \operatorname{int} S$.
- not intensive: $\operatorname{int} S \subseteq S$ doesn't always hold.

Interior operator

From now on we shall refer to subsets of L that are closed under meets as meet-subsets of L. The system of all meet-subsets in L will be denoted by

$$\mathsf{M}(L).$$

Interior operator

From now on we shall refer to subsets of L that are closed under meets as meet-subsets of L. The system of all meet-subsets in L will be denoted by

$$\mathsf{M}(L).$$

Proposition

For every $S \in \mathsf{M}(L)$:

- $\operatorname{int} S \subseteq S$.
- $\mathfrak{o}(a) \subseteq S$ *iff* $\mathfrak{o}(a) \subseteq \operatorname{int} S$.

Lemma

Let $f \colon L \to M$ be a meet-preserving map between locales and let $S \in \mathsf{M}(L)$ and $T \in \mathsf{M}(M)$. Then $f[S] \in \mathsf{M}(M)$ and $f^{-1}[T] \in \mathsf{M}(L)$ and we have again an adjunction

$$\mathsf{M}(L) \underset{f^{-1}[-]}{\overset{f[-]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{M}(M).$$

Lemma

Let $f: L \to M$ be a meet-preserving map between locales. Then $f^{-1}[\mathsf{O}] = \mathsf{O}$ if and only if $f^{-1}[\mathfrak{c}(b)] \subseteq (\mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]))^{\mathsf{c}}$ for every $b \in M$.

Localic Maps

A map $f\colon L \to M$ is a localic map if and only if the following conditions hold:

(a) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(b) $f^{-1}[\mathsf{O}] = \mathsf{O}$.

(c) $f^{-1}[\mathfrak{o}(b)] \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathsf{c}}$ for every $b \in M$.

Localic Maps

A map $f\colon L \to M$ is a localic map if and only if the following conditions hold:

(a) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(b) $f^{-1}[\mathrm{O}] = \mathrm{O}$.

(c) $f^{-1}[\mathfrak{o}(b)] \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathrm{c}}$ for every $b \in M$.

Using the previous lemma and the fact that $f^{-1}[\mathfrak{o}(b)]$ is a meet-subset we conclude that a map $f\colon L \to M$ is a localic map if and only if :

Localic Maps

A map $f\colon L \to M$ is a localic map if and only if the following conditions hold:

(a) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(b) $f^{-1}[\mathsf{O}] = \mathsf{O}$.

(c) $f^{-1}[\mathfrak{o}(b)] \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathsf{c}}$ for every $b \in M$.

Using the previous lemma and the fact that $f^{-1}[\mathfrak{o}(b)]$ is a meet-subset we conclude that a map $f\colon L \to M$ is a localic map if and only if :

(I) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

Localic Maps

A map $f\colon L \to M$ is a localic map if and only if the following conditions hold:

(a) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(b) $f^{-1}[\mathsf{O}] = \mathsf{O}$.

(c) $f^{-1}[\mathfrak{o}(b)] \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathsf{c}}$ for every $b \in M$.

Using the previous lemma and the fact that $f^{-1}[\mathfrak{o}(b)]$ is a meet-subset we conclude that a map $f\colon L \to M$ is a localic map if and only if :

(I) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(II) $f^{-1}[\mathfrak{c}(b)] \subseteq (\operatorname{int}(f^{-1}[\mathfrak{o}(b)]))^{\mathsf{c}}$.

Localic Maps

A map $f\colon L \to M$ is a localic map if and only if the following conditions hold:

(a) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(b) $f^{-1}[\mathsf{O}] = \mathsf{O}$.

(c) $f^{-1}[\mathfrak{o}(b)] \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathsf{c}}$ for every $b \in M$.

Using the previous lemma and the fact that $f^{-1}[\mathfrak{o}(b)]$ is a meet-subset we conclude that a map $f\colon L \to M$ is a localic map if and only if :

(I) $f^{-1}[\mathfrak{c}(b)]$ is closed for every $b \in M$.

(II) $f^{-1}[\mathfrak{c}(b)] \subseteq (\mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]))^{\mathsf{c}}$.

(III) $\mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]) \supseteq f^{-1}[\mathfrak{c}(b)]^{\mathsf{c}}$ for every $b \in M$.

Localic Maps

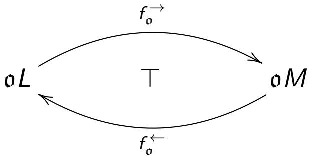
Proposition

A plain map $f: L \to M$ between locales is a localic map if and only if

$$(\mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]))^{\mathsf{c}} = f^{-1}[\mathfrak{c}(b)] \quad \textit{for every } b \in M.$$

Adjoint pair II

We now consider, for each (plain) map $f\colon L \to M$ between locales, the mappings $f_{\mathfrak{o}}^{\rightarrow}$ and $f_{\mathfrak{o}}^{\leftarrow}$ given by



$$\mathfrak{o}(a) \longmapsto (\mathrm{cl}\,(f[\mathfrak{c}(a)]))^{\mathsf{c}}$$

$$\mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]) \longmapsfrom \mathfrak{o}(b)$$

Adjoint pair II

Adjoint pair II

Theorem

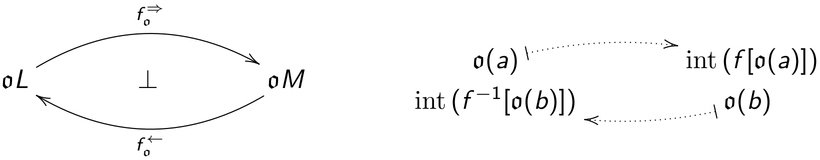
Let $f : L \to M$ be an order-preserving map between locales. The pair $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair if and only if f is a localic map.

Open Maps

We will speak about open maps in a broad sense as plain maps $f\colon L \to M$ between locales such that the image $f[\mathfrak{o}(a)]$ of every open sublocale is still open.

Adjoint pair III

As another variant, replace $f_{\mathfrak{o}}^{\rightarrow}$ by the following $f_{\mathfrak{o}}^{\Rightarrow}$:



Adjoint pair III

Adjoint pair III

Theorem

Let $f: L \to M$ be a meet-preserving map. The pair $(f_{\mathfrak{o}}^{\Rightarrow}, f_{\mathfrak{o}}^{\leftarrow})$ is an adjunction if and only if f is open.

Combining adjunctions II and III we get:

Corollary

An order-preserving map $f\colon L \to M$ is an open localic map if and only if

$$f_{\mathfrak{o}}^{\Rightarrow} \dashv f_{\mathfrak{o}}^{\leftarrow} \dashv f_{\mathfrak{o}}^{\rightarrow}.$$

Adjoint pair IV

Finally, consider the mappings

$$\mathfrak{c}L \underset{f_{\mathfrak{c}}^{\leftarrow}}{\overset{f_{\mathfrak{c}}^{\Rightarrow}}{\rightleftarrows}} \mathfrak{c}M \qquad (\top)$$

$$\mathfrak{c}(a) \longmapsto (\operatorname{int}(f[\mathfrak{o}(a)]))^{\mathsf{c}}$$

$$\operatorname{cl}(f^{-1}[\mathfrak{c}(b)]) \longleftarrow\!\!\dashv \mathfrak{c}(b)$$

Adjoint pair IV

Adjoint pair IV

Theorem

Let $f: L \to M$ be a meet-preserving map between locales. The pair $(f_{\mathfrak{c}}^{\leftarrow}, f_{\mathfrak{c}}^{\Rightarrow})$ is an adjunction if and only if f is an open localic map.

Open localic maps

Combining adjunctions I and IV we obtain:

Corollary

A plain map $f\colon L \to M$ is an open localic map if and only if

$$f_{\mathfrak{c}}^{\rightarrow} \dashv f_{\mathfrak{c}}^{\leftarrow} \dashv f_{\mathfrak{c}}^{\Rightarrow}.$$